This is the printout of a MathCad file (v.2001i). If you have this or a later version of MathCad, then click in this box to open the file.

Denominator:

$$D(x) := x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160$$

Factor:

$$\left(x^2 + 2\cdot x + 10\right)\cdot(x-4)^2$$

Numerator:

$$N(x) := x^4 - 6x^3 + 15x + 1$$

Long division:

$$\frac{x^4 - 6x^3 + 15x + 1}{x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160}$$

$$\text{Highest\_term\_num}(x) := x^4$$

$$\text{Highest\_term\_den}(x) := x^4$$

$$N(x) - \frac{\text{Highest\_term\_num}(x)}{\text{Highest\_term\_den}(x)}\cdot D(x) \rightarrow 63\cdot x - 159 - 10\cdot x^2$$

$$\frac{\text{Highest\_term\_num}(x)}{\text{Highest\_term\_den}(x)} \rightarrow 1$$

$$N(x) = 1\cdot D(x) + \left(63\cdot x - 159 - 10\cdot x^2\right)$$

Divide both sides by the denominator $D(x)$ and get:

$$\frac{N(x)}{D(x)} = \frac{D(x)}{D(x)} + \frac{63\cdot x - 159 - 10\cdot x^2}{x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160}$$

So now we have decomposed the original function into 1 plus a proper rational function.

Next we need to factor the denominator. In the hope that there is at least one rational x-intercept - which then has to be integer, as all the coefficients are -, we check all 24 divisors of the constant 160 :

$i := 0..1 \quad j := 0..5 \quad k := 0..3 \quad \text{divisors}_{i,j} := 2^j\cdot 5^i \quad \text{divisors}_{i+2,j} := -\text{divisors}_{i,j}$

$$\text{divisors} \rightarrow \begin{pmatrix} 1 & 2 & 4 & 8 & 16 & 32 \\ 5 & 10 & 20 & 40 & 80 & 160 \\ -1 & -2 & -4 & -8 & -16 & -32 \\ -5 & -10 & -20 & -40 & -80 & -160 \end{pmatrix}$$

$$\text{checkforroots}_{k,j} := D\left(\text{divisors}_{k,j}\right) \rightarrow \begin{pmatrix} 117 & 72 & 0 & 1440 & 42912 & 860832 \\ 45 & 4680 & 115200 & 2190240 & 37948320 & 631032480 \\ 225 & 360 & 1152 & 8352 & 93600 & 1257120 \\ 2025 & 17640 & 213120 & 2962080 & 44100000 & 680199840 \end{pmatrix}$$

So, $x = 4$ is an x-intercept of $D(x)$, therefore $x - 4$ is a factor of $D(x)$:

$$\frac{D(x)}{x-4} \rightarrow \frac{x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160}{x-4}$$

$$x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160 = (x-4)\cdot\left(x^3 - 2\cdot x^2 + 2\cdot x - 40\right)$$

Similarly, we find that $x - 4$ is also a factor of $\left(x^3 - 2\cdot x^2 + 2\cdot x - 40\right)$:

$$x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160 = (x-4)\cdot\left(x^3 - 2\cdot x^2 + 2\cdot x - 40\right) = (x-4)^2\cdot\left(x^2 + 2\cdot x + 10\right)$$

The last factor is easily seen to have no roots, so the decomposition of the denominator into irreducible factors gives us the following for the original function:

$$\frac{N(x)}{D(x)} = 1 + \frac{63\cdot x - 159 - 10\cdot x^2}{(x-4)^2\cdot\left(x^2 + 2\cdot x + 10\right)}$$

We need to find four constants A, B, C, and D ,such that the following identity holds

$$\frac{63\cdot x - 159 - 10\cdot x^2}{x^4 - 6\cdot x^3 + 10\cdot x^2 - 48\cdot x + 160} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C\cdot x + D}{x^2 + 2\cdot x + 10}$$

This is equivalent to (common denominator):

$$63\cdot x - 159 - 10\cdot x^2 = A\cdot(x-4)\cdot\left(x^2 + 2\cdot x + 10\right) + B\cdot\left(x^2 + 2\cdot x + 10\right) + (C\cdot x + D)\cdot(x-4)^2$$

If you expand the right side, you get a polynomial of degree 3 or less that is supposed to be identical to the polynomial on the left (the numerator), and so you get 4 equations for its coefficients.

$$A\cdot x^3 - 2\cdot A\cdot x^2 + 2\cdot A\cdot x - 40\cdot A + B\cdot x^2 + 2\cdot B\cdot x + 10\cdot B + C\cdot x^3 - 8\cdot C\cdot x^2 + 16\cdot C\cdot x + D\cdot x^2 - 8\cdot D\cdot x +$$

$$(C + A)\cdot x^3 + (B - 8\cdot C - 2\cdot A + D)\cdot x^2 + (2\cdot B + 16\cdot C + 2\cdot A - 8\cdot D)\cdot x + 16\cdot D + 10\cdot B - 40\cdot A$$

Here are the 4 (linear) equations:

Given

$$C + A = 0$$

$$B - 8\cdot C - 2\cdot A + D = -10$$

$$2\cdot B + 16\cdot C + 2\cdot A - 8\cdot D = 63$$

$$16\cdot D + 10\cdot B - 40\cdot A = -159$$

$$\text{Find}(A,B,C,D) \rightarrow \begin{pmatrix} \frac{23}{289} \\ \frac{-67}{34} \\ \frac{-23}{289} \\ \frac{-4917}{578} \end{pmatrix}$$

Comment:

This is a linear system. Your calculator can solve it in matrix form:

$$M := \begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & -8 & 1 \\ 2 & 2 & 16 & -8 \\ -40 & 10 & 0 & 16 \end{pmatrix} \qquad R := \begin{pmatrix} 0 \\ -10 \\ 63 \\ -159 \end{pmatrix} \qquad M^{-1}\cdot R \rightarrow \begin{pmatrix} \frac{23}{289} \\ \frac{-67}{34} \\ \frac{-23}{289} \\ \frac{-4917}{578} \end{pmatrix} = \begin{pmatrix} 0.08 \\ -1.971 \\ -0.08 \\ -8.507 \end{pmatrix}$$

$$S := M^{-1}\cdot R$$

$$\frac{N(x)}{D(x)} = 1 + \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C\cdot x + D}{x^2 + 2\cdot x + 10} = 1 + \frac{\frac{23}{289}}{x-4} + \frac{\frac{-67}{34}}{(x-4)^2} + \frac{\frac{-23}{289}\cdot x + \frac{-4917}{578}}{x^2 + 2\cdot x + 10} = \blacksquare$$

$$\blacksquare = 1 + \frac{1}{578}\cdot\left[\left[\frac{46}{x-4} + \frac{-1139}{(x-4)^2} + \frac{-46\cdot x - 4917}{x^2 + 2\cdot x + 10}\right]\right] = \blacksquare$$

$$\blacksquare = 1 + \frac{1}{578}\cdot\left[\left[\frac{46}{x-4} + \frac{-1139}{(x-4)^2} + \frac{-46\cdot x - 46}{x^2 + 2\cdot x + 10} + \frac{46 - 4917}{x^2 + 2\cdot x + 10}\right]\right]$$

Now you can integrate: the third expression in parentheses turns into a multiple of the logarithm of its denominator, and the fourth is an arctangent.