Interpolation and Lagrange Polynomials - (3.1)

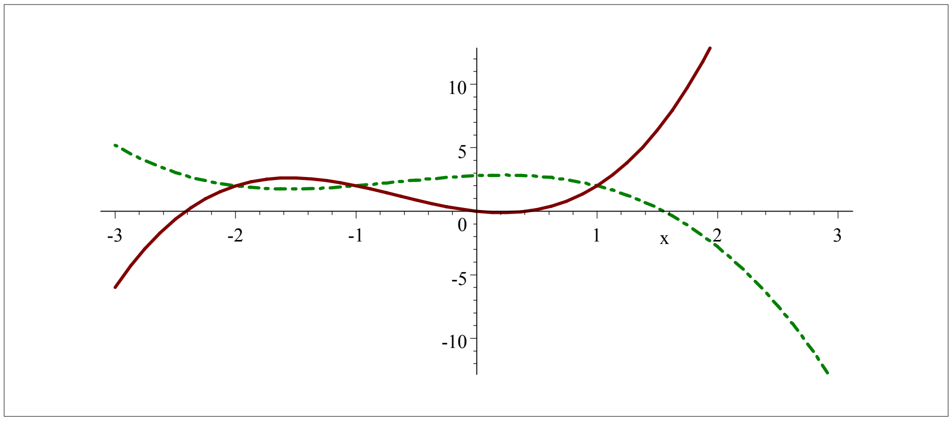
1. Polynomial Interpolation:

Problem: Given $n+1$ pairs of data points $\left(x_i,\ y_i\right)$, $i = 0,1,\ldots,n$, we want to find a polynomial $P_k(x)$ of **lowest** possible degree for which $P_k(x_i) = y_i$, $i = 0,1,\ldots,n$.

The polynomial $P_k(x)$ is said to **interpolate** the data $\left(x_i,\ y_i\right)$, $i = 0,1,\ldots,n$ and is called an **interpolating polynomial.**

Graphically, $P_k(x)$ is an **approximation** to $f(x)$ and **satisfies the conditions**:

$$P_k(x_i) = f(x_i), \quad i = 0,1,\ldots,n.$$



$$— \ y = f(x), \quad \text{-.-.-}\ y = P_k(x)$$

Obviously, for this example $P_k(x)$ is not a good approximation to $f(x)$ though $P_k(x)$ satisfies the conditions:

$$P_k(x_i) = y_i \quad \text{for } i = 0,1,2.$$

Note that the differences between a kth degree **Taylor polynomial** and a kth degree **interpolating polynomial** are:

- **a.** $P_{\text{Taylor}}(x) = f(x)$ at only $x = x_0$ and $P_{\text{interpolating}}(x) = f(x)$, at $x = x_0,\ x_1,\ \ldots,\ x_n$.
- **b.** P_{Taylor} requires knowledge of $f',\ f'',\ \ldots$ but $P_{\text{interpolating}}$ requires $f(x_0),\ f(x_1),\ \ldots,\ f(x_n)$.

2. Lagrange Interpolating Polynomials:

a. Lagrange Polynomials:

For $k = 0,1,\ldots,n$, define

$$L_{n,k}(x) = \prod_{i=0,\ i\neq k}^{n} \frac{(x-x_i)}{(x_k-x_i)} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}.$$

$L_{n,k}(x)$ are called **Lagrange polynomials**. For example, let $x_i = i$, $i = 0,1,2,3$.

$$L_{3,1}(x) = \frac{x(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3)$$

Observe that Lagrange Polynomials have the following **properties:**

i. $L_{n,k}(x_k) = 1$

$$L_{n,k}(x_k) = \frac{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)} = 1$$

ii. $L_{n,k}(x_i) = 0$ for $i \neq k$.

$$L_{n,k}(x_i) = \frac{(x_i-x_0)\ldots(x_i-x_i)\ldots(x_i-x_{k-1})(x_i-x_{k+1})\ldots(x_i-x_n)}{(x_k-x_0)\ldots(x_k-x_i)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)} = 0$$

b. Lagrange Interpolating Polynomials:

The polynomial

$$P_n(x) = \sum_{k=0}^{n} y_k \, L_{n,k}(x) = y_0 \, L_{n,0}(x) + y_1 \, L_{n,1}(x) + \ldots + y_n \, L_{n,n}(x)$$

is called the nth **Lagrange interpolating polynomial**. Observe that for $i = 0, 1, \ldots, n$

$$P_n(x_i) = \sum_{k=0}^{n} y_k \, L_{n,k}(x_i) = y_i \, L_{n,i}(x_i) = y_i.$$

$P_n(x)$ is an nth polynomial that agrees with $f(x)$ at $x_0, \; x_1, \; \ldots, \; x_n$.

Theorem **3.3** *Suppose that* $x_0, \; \ldots, \; x_n$ *are* ***distinct numbers*** *in the interval* $[a, \; b]$ *and* $f^{(n+1)}$ *is* ***continuous*** *in* $[a, \; b]$*. For each* x *in* $[a, \; b]$*, there exists a number* $c(x)$ *in* $(a, \; b)$ *such that*

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c(x))}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

Proof For $x \neq x_i$ and x in $[a, \; b]$, define $g(t) = f(t) - P_n(t) - (f(x) - P_n(x)) \prod_{k=0}^{n} \frac{(t - x_k)}{x - x_k}$. Clearly, $g(x_i) = 0, \; i = 0, 1, \ldots, n$. Observe that $g(x) = 0$. Since $x \neq x_i$,

$$g(t) = 0 \text{ at } t = x_0, \; x_1, \; \ldots, \; x_n, \; x.$$

Since $g(t)$ has $n + 2$ distinct zeros in $[a, \; b]$, by Rolle's Theorem, we know there exists a constant c in $(a, \; b)$ such that $g^{(n+1)}(c) = 0$.

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (f(x) - P_n(x))\frac{(n+1)!}{(x - x_0)(x - x_1)\ldots(x - x_n)}$$

$$g^{(n+1)}(c) = f^{(n+1)}(c) - 0 - (f(x) - P_n(x))\frac{(n+1)!}{(x - x_0)(x - x_1)\ldots(x - x_n)} = 0$$

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)(c)}}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

Comparing the error term for an nth Lagrange polynomial with the error term for an *nth* Taylor polynomial.

$$R_n^{\text{Taylor}}(x) = f(x) - P_n(x) = \frac{f^{(n+1)(c)}}{(n+1)!}(x - x_0)^n$$

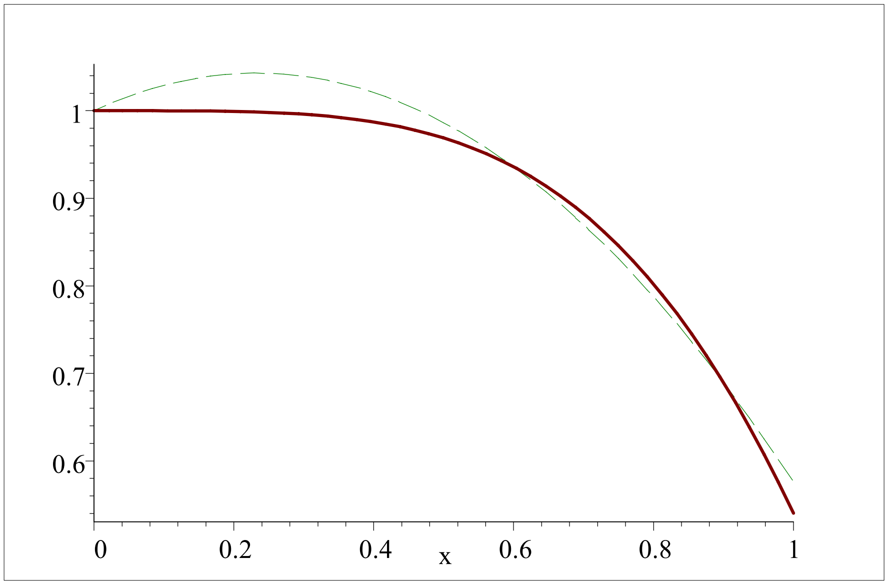
Example *Let* $x_0 = 0, \; x_1 = 0.6, \; x_2 = 0.9, \; f(x) = cos(x^2)$.

- **i.** *Find the Lagrange interpolating polynomial* $P_2(x)$ *and* $R_2(x)$.
- **ii.** *Use* $P_2(x)$ *to approximate* $f(0.45)$ *and estimate the approximation error.*
- **iii.** *Approximate* $\int_0^1 cos(x^2)dx$ *by* $\int_0^1 P_2(x)dx$ *and estimate the approximation error.*

i.

$$P_2(x) = f(0)\frac{(x - 0.6)(x - 0.9)}{(-0.6)(-0.9)} + f(0.6)\frac{x(x - 0.9)}{0.6(-0.3)} + f(0.9)\frac{x(x - 0.6)}{0.9(0.3)}$$

$$P_2(x) = \frac{1}{0.54}(x - 0.6)(x - 0.9) - \frac{\cos(0.36)}{0.18}x(x - 0.9) + \frac{\cos(0.81)}{0.27}x(x - 0.6)$$



— $y = \cos(x^2)$, - - - $y = P_2(x)$

$$f'(x) = -2x\sin(x^2), \quad f''(x) = -2(\sin(x^2) + 2x^2\cos(x^2)),$$

$$f'''(x) = -2(2x\cos(x^2) + 4x\cos(x^2) - 4x^3\sin(x^2)) = -4(3x\cos(x^2) - 2x^3\sin(x^2))$$

$$|\,R_2(x)\,| = \frac{-4(3x\cos(c) - 2c^3\sin(c))}{6}x(x-0.6)(x-0.9)$$

where c is in $(0,\ 1)$.

ii.

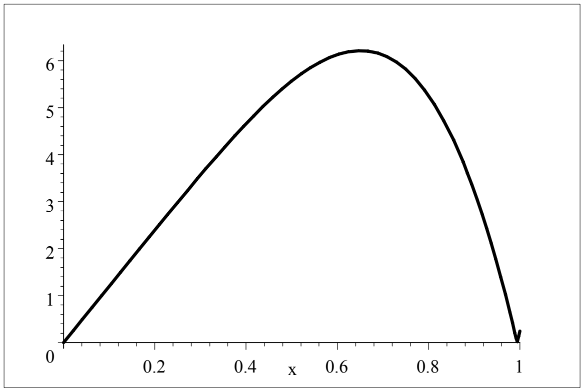
$$\begin{aligned} f(0.45) &\approx P_2(0.45) \\ &= \frac{1}{0.54}(-0.15)(-0.45) - \frac{\cos(0.36)}{0.18}(0.45)(-0.45) + \frac{\cos(0.81)}{0.27}(0.45)(-0.15) \\ &= 1.005509 \end{aligned}$$

True error $= |\ \cos\left((0.45)^2\right) - P_2(0.45)| = |\ 0.9795668 - 1.005509| = 0.0259422$

Find the approximation error:

$$|\ f'''(x)| \le \max_{x \text{ in } (0,1)} |\ -4(3x\cos(x^2) - 2x^3\sin(x^2))| \le 4(3+2) = 20$$

or check out the graph of $|f'''(x)|$ to have a better estimate of an upper bound of $|\ f'''(x)|$:



$y = |-4(3x\cos(x^2) - 2x^3\sin(x^2))|$

$|\ f'''(x)| \le 6.5$, for x in $(0,\ 1)$

$$\begin{aligned} |\ R_2(0.45)| &\le \left|\frac{6.5}{6}0.45(-0.15)(-0.45)\right| \\ &= 0.03290 \end{aligned}$$

iii.

$$\begin{aligned}\int_0^1 P_2(x)dx &= \int_0^1\left(\frac{1}{0.54}(x-0.6)(x-0.9)-\frac{\cos(0.36)}{0.18}x(x-0.9)+\frac{\cos(0.81)}{0.27}x(x-0.6)\right)dx\\ &= \frac{1}{0.54}\int_0^1\big((x-0.6)^2-0.3(x-0.6)\big)dx-\frac{\cos(0.36)}{0.18}\int_0^1(x^2-0.9x)dx\\ &\quad+\frac{\cos(0.81)}{0.27}\int_0^1(x^2-0.6x)dx\\ &= \frac{1}{0.54}\left(\frac{1}{3}(x-0.6)^3-\frac{0.3}{2}(x-0.6)^2\right)\Big|_0^1-\frac{\cos(0.36)}{0.18}\left(\frac{1}{3}x^3-\frac{0.9}{2}x^2\right)\Big|_0^1\\ &\quad+\frac{\cos(0.81)}{0.27}\left(\frac{1}{3}x^3-\frac{0.6}{2}x^2\right)\Big|_0^1\\ &= \frac{1}{0.54}\left(\frac{1}{3}(0.4)^3-\frac{0.3}{2}(0.4)^2+\frac{1}{3}(0.6)^3+\frac{0.3}{2}(0.6)^2\right)-\frac{\cos(0.36)}{0.18}\left(\frac{1}{3}-\frac{0.9}{2}\right)\\ &\quad+\frac{\cos(0.81)}{0.27}\left(\frac{1}{3}-\frac{0.6}{2}\right)\\ &= 0.9201181\end{aligned}$$

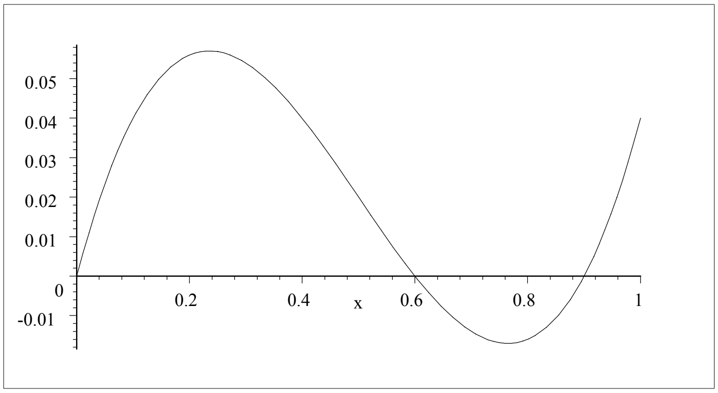
True error:

$$\left|\int_0^1\cos(x^2)dx-\int_0^1 P_2(x)dx\right| = \left|\ 0.9045242-0.9201181\right| = 0.0155939$$

An approximation error:

$$\begin{aligned}\text{Error}_{\text{approx}} &= \left|\int_0^1\cos(x^2)dx-\int_0^1 P_2(x)dx\right| \le \int_0^1\left|\ R_2(x)\right|dx\\ &= \int_0^1\frac{\left|f'''(c)\right|}{3!}\left|\ x(x-0.6)(x-0.9)\right|dx\end{aligned}$$

Observe that $\left|\ f'''(\xi(x))\right| \le 6.5$, and the function $x(x-0.6)(x-0.9)$ changes signs over $\left[0,\ 1\right]$:



$$y = x(x-0.6)(x-0.9),\ 0\le x\le 1$$

$$\begin{aligned}x(x-0.6)(x-0.9) &= (x-0.6+0.6)(x-0.6)(x-0.6-0.3)\\ &= \left[(x-0.6)^2+0.6(x-0.6)\right](x-0.6-0.3)\\ &= (x-0.6)^3+0.3(x-0.6)^2-0.18(x-0.6)\end{aligned}$$

$$\begin{aligned}x(x-0.6)(x-0.9) &= (x-0.9+0.9)(x-0.9+0.3)(x-0.9)\\ &= \big((x-0.9)^2+1.2(x-0.9)+0.27\big)(x-0.9)\\ &= (x-0.9)^3+1.2(x-0.9)^2+0.27(x-0.9)\end{aligned}$$

$$
\begin{aligned}
\text{Error}_{\text{approx}} &\le \frac{6.5}{6}\left(\int_0^{0.6} x(x-0.6)(x-0.9)dx - \int_{0.6}^{0.9} x(x-0.6)(x-0.9)dx + \int_{0.9}^{1} x(x-0.6)(x-0.9)dx\right) \\
&= \left[\frac{1}{4}(x-0.6)^4 + 0.1(x-0.6)^3 - 0.09(x-0.6)^2\right]\Big|_0^{0.6} \\
&\quad - \left[\frac{1}{4}(x-0.6)^4 + 0.1(x-0.6)^3 - 0.09(x-0.6)^2\right]\Big|_{0.6}^{0.9} \\
&\quad + \left[\frac{1}{4}(x-0.9)^4 + 0.4(x-0.9)^3 + \frac{0.27}{2}(x-0.9)^2\right]\Big|_{0.9}^{1} \\
&= \frac{1}{4}(0.6)^4 + 0.1(0.6)^3 - 0.09(0.6)^2 - \frac{1}{4}(0.3)^4 - 0.1(0.3)^3 + 0.09(0.3)^2 \\
&\quad + \frac{1}{4}(0.1)^4 + 0.4(0.1)^3 + \frac{0.27}{2}(0.1)^2 \\
&= 0.0268979
\end{aligned}
$$

3. Neville's Iterated Interpolation:

Naturally if $P_n(x)$ is not a good approximation to $f(x)$ we like to **add** more points $x_{n+1}, \ldots$ to construct an interpolation polynomial with **larger** degree. Can we construct an interpolation polynomial **iteratively**? The answer is yes. Neville's Method does so.

Let $m_1, m_2, \ldots, m_k$ be k distinct integers where $0 \le m_i \le n$ for each i. Let $P_{m_1,m_2\ldots,m_k}(x)$ be the Lagrange polynomial that agrees with f at the k points: $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$. For example,

$$P_{1,2,4}(x) = f(x_1)\frac{(x-x_2)(x-x_4)}{(x_1-x_2)(x_1-x_4)} + f(x_2)\frac{(x-x_1)(x-x_4)}{(x_2-x_1)(x_2-x_4)} + f(x_4)\frac{(x-x_1)(x-x_2)}{(x_4-x_1)(x_4-x_2)}$$

is the Lagrange polynomial that **agrees** with $f(x)$ at 3 points: x_1, x_2, x_4 :

$$P_{1,2,4}(x_1) = f(x_1), \quad P_{1,2,4}(x_2) = f(x_2), \quad P_{1,2,4}(x_3) = f(x_3).$$

Theorem 3.5 *Let f be defined at $x_0, x_1, \ldots, x_k$, and x_j and x_k be two distinct numbers. Then*

$$P_{0,1,\ldots,k}(x) = \frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{x_i - x_j}.$$

Proof We know that both polynomials $P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)$ and $P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)$ are degree $k-1$. So, the degree of $P_{0,1,\ldots,k}(x)$ is k. Check if $P_{0,1,\ldots,k}(x_l) = f(x_l)$ for $l = 0, 1, \ldots, k$: since

$$P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_l) = f(x_l) \quad \text{if } l \ne j \quad \text{and} \quad P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_l) = f(x_l) \quad \text{if } l \ne i$$

For $l = 0, 1, \ldots, k$, but $l \ne i$ and $l \ne j$,

$$
\begin{aligned}
P_{0,1,\ldots,k}(x_l) &= \frac{(x_l-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_l) - (x_l-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_l)}{x_i - x_j} \\
&= \frac{(x_l-x_j)f(x_l) - (x_l-x_i)f(x_l)}{(x_i-x_j)} = \frac{(x_i-x_j)}{(x_i-x_j)}f(x_l) = f(x_l)
\end{aligned}
$$

For $l = i$:

$$
\begin{aligned}
P_{0,1,\ldots,k}(x_i) &= \frac{(x_i-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_i) - (x_i-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_i)}{x_i - x_j} \\
&= P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_i) = f(x_i)
\end{aligned}
$$

For $l = j$:

$$
\begin{aligned}
P_{0,1,\ldots,k}(x_i) &= \frac{(x_j-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x_j) - (x_j-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_j)}{x_i - x_j} \\
&= P_{0,1,\ldots,i-1,i+1,\ldots,k}(x_j) = f(x_j)
\end{aligned}
$$

So, $P_{0,1,\ldots,k}(x)$ the kth degree interpolating polynomial.

Example *Consider* $x_0,\ x_1, \ldots,\ x_n$. *Then*

$$P_{1,2}(x) = f(x_1)\frac{(x-x_2)}{(x_1-x_2)} + f(x_2)\frac{(x-x_1)}{(x_2-x_1)}$$

$$P_{2,4}(x) = f(x_2)\frac{(x-x_4)}{(x_1-x_4)} + f(x_4)\frac{(x-x_4)}{(x_2-x_4)}$$

$$P_{1,2,4}(x) = \frac{(x-x_4)P_{1,2}(x) - (x-x_1)P_{2,4}(x)}{(x_1-x_4)}$$

is the Lagrange polynomial that agrees with $f(x)$ *at* 3 *points:* $x_1,\ x_2,\ x_4$.

Neville's Iterated Interpolation: Steps of Computing $P_{0,1,2,\ldots,k}(\bar{x})$:

Given $(x_i,\ f(x_i))$ for $i = 0, 1, \ldots, k$ and ε, let $P_i = f(x_i)$

x_i	$P_i(\bar{x})$	$P_{i,i+1}(\bar{x})$	$P_{i,i+1,i+2}(\bar{x})$	$P_{i,i+1,i+2,i+3}(\bar{x})$	$P_{i,i+1,i+2,i+3,i+4}(\bar{x})$
x_0	P_0				
x_1	P_1	$P_{0,1}(\bar{x})$			
x_2	P_2	$P_{1,2}(\bar{x})$	$P_{0,1,2}(\bar{x})$		
x_3	P_3	$P_{2,3}(\bar{x})$	$P_{1,2,3}(\bar{x})$	$P_{0,1,2,3}(\bar{x})$	
x_4	P_4	$P_{3,4}(\bar{x})$	$P_{2,3,4}(\bar{x})$	$P_{1,2,3,4}(\bar{x})$	$P_{0,1,2,3,4}(\bar{x})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The algorithm stops and $f(\bar{x}) \approx P_{0,1,2,\ldots,k}(\bar{x})$ whenever

$$\left| P_{0,1,2,\ldots,k}(\bar{x}) - P_{0,1,2,\ldots,k-1}(\bar{x}) \right| < \varepsilon.$$

An alternative way:

x_i	$P_i(\bar{x})$	$P_{i,i+1}(\bar{x})$	$P_{i,i+1,i+2}(\bar{x})$	$P_{i,i+1,i+2,i+3}(\bar{x})$	$P_{i,i+1,i+2,i+3,i+4}(\bar{x})$
x_0	P_0				
x_1	P_1	$P_{0,1}(\bar{x})$			
x_2	P_2	$P_{0,2}(\bar{x})$	$P_{0,1,2}(\bar{x})$		
x_3	P_3	$P_{0,3}(\bar{x})$	$P_{0,2,3}(\bar{x})$	$P_{0,1,2,3}(\bar{x})$	
x_4	P_4	$P_{0,4}(\bar{x})$	$P_{0,3,4}(\bar{x})$	$P_{0,2,3,4}(\bar{x})$	$P_{0,1,2,3,4}(\bar{x})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Example *Suppose that* $x_j = j$ *for* $j = 0, 1, 2, 3$ *and it is known that*

$$P_{0,1}(x) = x + 1, \qquad P_{1,2}(x) = 3x - 1, \qquad \textit{and} \qquad P_{1,2,3}(1.5) = 4.$$

Find $P_{0,1,2,3,}(1.5)$.

x_i	$P_i(\bar{x})$	$P_{i,i+1}(\bar{x})$	$P_{i,i+1,i+2}(\bar{x})$	$P_{i,i+1,i+2,i+3}(\bar{x})$
0	P_0			
1	P_1	$P_{0,1}(1.5) = 2.5$		
2	P_2	$P_{1,2}(1.5) = 3.5$	$P_{0,1,2}(1.5) = 3.25$	
3	P_3	$P_{0,3}(1.5)$	$P_{1,2,3}(1.5) = 4$	$P_{0,1,2,3}(1.5) = 5.4375$

$$P_{0,1,2}(1.5) = \frac{(1.5-2)P_{0,1}(1.5) - (1.5-0)P_{1,2}(1.5)}{(0-2)} = \frac{(-0.5)(2.5) - (1.5)(3.5)}{(-2)} = 3.25$$

$$P_{0,1,2,3}(1.5) = \frac{(1.5-3)P_{0,1,2}(1.5) - (1.5-0)P_{1,2,3}(1.5)}{(0-3)} = \frac{(-1.5)(3.25) - (1.5)(4)}{(-2)} = 5.4375$$

Example *Neville's method is used to approximate f*(0.4) *as follows. Complete the table.*

x_i	$P_i(\bar{x})$	$P_{i,i+1}(\bar{x})$	$P_{i,i+1,i+2}(\bar{x})$	$P_{i,i+1,i+2,i+3}(\bar{x})$
0	1			
0.25	2	$P_{0,1}(0.4) = 2.6$		
0.5	P_2	$P_{1,2}(0.4)$	$P_{0,1,2}(0.4)$	
0.75	8	$P_{2,3}(0.4) = 2.4$	$P_{1,2,3}(0.4) = 2.96$	$P_{0,1,2,3}(0.4) = 3.016$

$$P_{2,3}(0.4) = \frac{(0.4-0.75)P_2 - (0.4-0.5)P_3}{0.5-0.75} = 2.4, \quad P_2 = \frac{(2.4)(-0.25) - 0.1(8)}{-0.35} = 4$$

$$P_{1,2}(0.4) = \frac{(0.4-0.5)P_1 - (0.4-0.25)P_2}{0.25-0.5} = \frac{(0.4-0.5)(2) - (0.4-0.25)(4)}{0.25-0.5} = 3.2$$

$$P_{0,1,2}(0.4) = \frac{(0.4-0.5)P_{0,1}(0.4) - (0.4-0)P_{1,2}(0.4)}{0-0.5} = \frac{(0.4-0.5)(2.6) - (0.4-0)(3.2)}{0-0.5} = 3.08$$

Check:

$$P_{0,1,2,3}(0.4) = \frac{(0.4-0.75)(3.08) - (0.4-0)(2.96)}{0-0.75} = 3.016$$

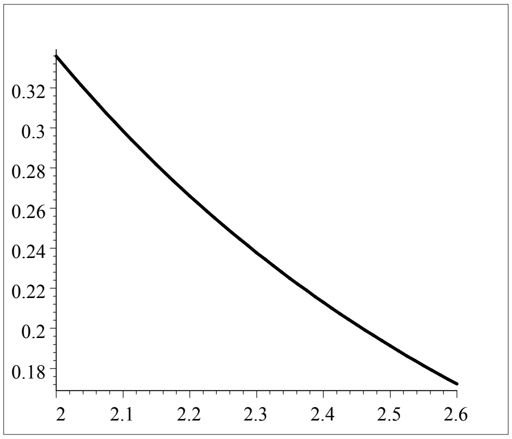
Example *Let f*$(x) = \sin(\ln x)$, $x_0 = 2.0$, $x_1 = 2.4$, $x_2 = 2.6$. *Find a bound for the absolute error on* $[2.0,\ 2.6]$. *Approximate f*(2.5).

$$|R_2(x)| = \left|\frac{f'''(c)}{3!}(x-2)(x-2.4)(x-2.6)\right|, \text{ where } c \text{ is in } [2,\ 2.6]$$

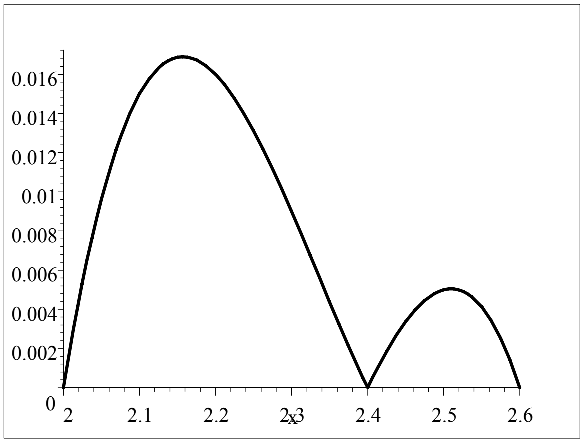
$$f'(x) = \frac{\cos(\ln x)}{x}, \quad f''(x) = \frac{-\sin(\ln x)(\frac{1}{x})(x) - \cos(\ln x)}{x^2} = \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$$

$$f'''(x) = \frac{[-\cos(\ln x)(\frac{1}{x}) + \sin(\ln x)(\frac{1}{x})]x^2 + 2x[\sin(\ln x) + \cos(\ln x)]}{x^4}$$

$$= \frac{[-\cos(\ln x) + \sin(\ln x)] + 2[\sin(\ln x) + \cos(\ln x)]}{x^3} = \frac{3\sin(\ln x) + \cos(\ln x)}{x^3}$$



$\left|f'''(x)\right|$



$y = |(x-2)(x-2.4)(x-2.6)|$

$$\left|f'''(x)\right| \le \left|f'''(2)\right| = \frac{3\sin(\ln 2) + \cos(\ln 2)}{2^3} = 0.336$$

$$\left|\ R_2(x)\right| = \left|\frac{f'''(c)}{3!}(x-2)(x-2.4)(x-2.6)\right| \le \frac{0.336}{6}(0.02) = 0.00112$$

```
>> xv=[2;2.4;2.6];
>> yv=sin(log(xv));
>> [yout,yall]=neville(xv,yv,2.5,3)
   yout =0.7935
    yall = 0.6390  0.8001     0.7935
              0.7678   0.7922 0     0
              0.8166   0                  0
```