

5.2 Solutions about Singular Points

Consider a second order linear differential equation of the form:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \quad (1)$$

The standard form of (1) is

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{where} \quad P(x) = \frac{a_1(x)}{a_2(x)}, \quad \text{and} \quad Q(x) = \frac{a_0(x)}{a_2(x)}. \quad (2)$$

The general solution of the equation in (1) is $y = C_1y_1 + C_2y_2$.

As described in the last Chapter, there exist **two linearly independent solutions** of (1) in the form of a power series centered at x_0 where x_0 is an **ordinary point** of the equation. In this section, we will look into to the possibility of finding a power series solution or a power series like solution centered at x_0 when x_0 is a **singular point** of the equation.

1. Regular and Irregular Singular Points:

Definition:

Let x_0 be a **singular point** of the equation (1). x_0 is said to be a **regular singular point** if both functions

$$(x - x_0)P(x) = (x - x_0) \frac{a_1(x)}{a_2(x)} \quad \text{and} \quad (x - x_0)^2Q(x) = (x - x_0)^2 \frac{a_0(x)}{a_2(x)}$$

are **analytic** at x_0 . x_0 is said to be an **irregular singular point** if it is not regular.

Example Find all singular points of the equation and for each singular point determine if it is regular singular or irregular singular.

a. $x(x - 1)y'' + y' + 2y = 0$

b. $x^2y'' - (2 + x^2)y' + xy = 0$

- a. $a_2(x) = x(x - 1) = 0$ when $x = 0$, $x = 1$, and $a_1(x) = 1$ and $a_0(x) = 2$ are polynomials. The differential equation has **2 singular points**: $x = 0$, and $x = 1$. Determine now if these singular points are regular or irregular.

- i. Consider $x = 0$:

$$xP(x) = x \frac{1}{x(x - 1)} = \frac{1}{x - 1}, \quad x^2Q(x) = x^2 \frac{2}{x(x - 1)} = \frac{2x}{x - 1}.$$

Since both $\frac{1}{x - 1}$ and $\frac{2x}{x - 1}$ are **analytic** at $x = 0$, $x = 0$ is a **regular singular point**.

- ii. Consider $x = 1$:

$$(x - 1)P(x) = (x - 1) \frac{1}{x(x - 1)} = \frac{1}{x}, \quad (x - 1)^2Q(x) = (x - 1)^2 \frac{2}{x(x - 1)} = \frac{2(x - 1)}{x}.$$

Since both $\frac{1}{x}$ and $\frac{2(x - 1)}{x}$ are **analytic** at $x = 1$, $x = 1$ is a **regular singular point**.

- b. $a_2(x) = x^2 = 0$ when $x = 0$, and $a_1(x) = -(2 + x^2)$ and $a_0(x) = x$ are polynomials. The differential equation has 1 singular point: $x = 0$. Determine now if this singular point is regular or irregular. Check

$$xP(x) = -x \frac{2 + x^2}{x^2} = -\frac{2 + x^2}{x}.$$

Since $xP(x)$ is **not analytic** at $x = 0$, $x = 0$ is an **irregular singular point**.

2. Frobenius' Theorem:

Let $x = x_0$ be a **regular singular point** of the equation (1). Then there exists **at least one** solution of the form

$$y = (x - x_0)^r \sum_{m=0}^{\infty} c_m (x - x_0)^m = \sum_{m=0}^{\infty} c_m (x - x_0)^{m+r}$$

where r is a constant to be determined so that $c_0 \neq 0$. There also exists a constant $R > 0$ such that the power series is convergent at least on the interval $0 < x - x_0 < R$.

Note that when x_0 is a regular singular point,

- **there may not be two linearly independent solutions** in the form of $\sum_{m=0}^{\infty} c_m (x - x_0)^{m+r}$;
- $\sum_{m=0}^{\infty} c_m (x - x_0)^{m+r}$ is **not a power series** if r is not a **nonnegative integer**.

Cauchy - Euler equations:

$$a^2 x^2 y'' + bxy' + cy = 0 \quad \text{where } a, b \text{ and } c \text{ are given constants}$$

is a special case. For example, the equation: $x^2 y'' - 3xy' + 4y = 0$ has two linearly independent solutions:

$$y_1 = x^2 \quad \text{and} \quad y_2 = x^2 \ln x.$$

Since $\ln x$ is not defined at $x = 0$, y_2 does not have a power series in the powers of x .

Three Possible Cases for the Solutions y_1 and y_2 :

i. $r_2 = r_1$.

Let $y_1 = \sum_{m=0}^{\infty} c_m x^{m+r_1}$ $c_0 \neq 0$ be a solution of the equation. Use the **method of reduction of order** to derive the second solution y_2 , Let $y_2 = u(x)y_1$. Then

$$u = \int \frac{1}{y_1^2} e^{-\int P(x) dx} dx$$

$$y_2 = u(x)y_1 = y_1 \int \frac{1}{y_1^2} \left(e^{-\int P(x) dx} \right) dx \quad (3)$$

$$y = C_1 y_1 + C_2 y_2.$$

Note that the solution y_2 can also be represented by a power series since $y_1, y_1^2, \frac{1}{y_1^2}, e^{-\int P(x) dx}$ are in power series form.

ii. $r_2 > r_1$ and $r_2 - r_1 \neq$ **an integer**. Then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} c_{m1} x^m = \sum_{m=0}^{\infty} c_{m1} x^{m+r_1}, \quad \text{where } c_{01} \neq 0,$$

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} c_{m2} x^m = \sum_{m=0}^{\infty} c_{m2} x^{m+r_2}, \quad \text{where } c_{02} \neq 0, \text{ and}$$

$$y = C_1 y_1 + C_2 y_2.$$

iii. $r_2 > r_1$ and $r_2 - r_1 = N$ **a positive integer**.

Let $y_1 = \sum_{m=0}^{\infty} c_{m1} x^{m+r_1}$ and $y_2 = \sum_{m=0}^{\infty} c_{m2} x^{m+r_2}$. There are **2 situations**:

(a) y_1 and y_2 are **linearly independent**, then $y = C_1 y_1 + C_2 y_2$.

(b) y_1 and y_2 are **linearly dependent**, or only one solution, say y_1 is derived, then compute y_2 by the formula in (3) and $y = C_1 y_1 + C_2 y_2$.

Frobenius' Method:

Let x_0 be a **regular singular point** of a given differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Find a **solution** of the equation of the form:

$$y = \sum_{m=0}^{\infty} c_m (x - x_0)^{m+r}.$$

Steps:

a. Compute

$$y' = \sum_{m=0}^{\infty} c_m (m+r)(x-x_0)^{m+r-1}, \quad y'' = \sum_{m=0}^{\infty} c_m (m+r)(m+r-1)(x-x_0)^{m+r-2}.$$

b. Substitute y , y' and y'' into the differential equation and simplify.

c. **Solve r so that $c_0 \neq 0$** and for each solved r find the recurrence relation for c_m and express c_m in terms of m if possible.

d. If there exists only one solution y_1 which is in the form of a power series, then compute y_2 :

$$y_2 = y_1 \int \frac{1}{y_1^2} \left(e^{-\int P(x) dx} \right) dx$$

The general solution $y = C_1 y_1 + C_2 y_2$.

Note that there are three main differences between Frobenius' Method and the Power Series Method:

- $x = x_0$ is a regular singular point of the equation.
- The starting indices of y , y' and y'' are 0.
- The value or values of r will need to be found.

The equation used to solve r is called **the indicial equation** and solutions r_1 and r_2 are called the **indicial roots**. The indicial equation and its root(s) can be solved at advance from the equation:

$$(r)(r-1) + a_0 r + b_0 = 0$$

where constants a_0 and b_0 are given in the series for $xP(x)$ and $x^2Q(x)$, respectively, i.e., :

$$(x-x_0)P(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots \text{ and}$$

$$(x-x_0)^2Q(x) = \sum_{m=0}^{\infty} b_m (x-x_0)^m = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \dots$$

Example Solve $3xy'' + y' - y = 0$ using Frobenius' Method.

$a_2(x) = 3x = 0$ when $x = 0$. Check the differential equation: $y'' + \frac{1}{3x} y' - \frac{1}{3x} y = 0$

$$xP(x) = x \left(\frac{1}{3x} \right) = \frac{1}{3} \quad \left(a_0 = \frac{1}{3} \right)$$

$$x^2Q(x) = x^2 \left(-\frac{1}{3x} \right) = -\frac{x}{3} = 0 - \frac{1}{3}x \quad \left(b_0 = 0 \right)$$

both are analytic at $x = 0$. So, $x = 0$ is a **regular singular point** of the equation. The indicial equation:

$$(r)(r-1) + \frac{1}{3}r = r \left(r - 1 + \frac{1}{3} \right) = r \left(r - \frac{2}{3} \right) = 0$$

and its roots are:

$$r = 0 \text{ or } r = \frac{2}{3}.$$

Since $r_2 - r_1 = \frac{2}{3}$ is not an integer, there exist **2 linearly independent solutions** in the form of a power series centered at 0, i.e.,

$$y_1 = \sum_{m=0}^{\infty} c_m x^{m+r_1} \quad \text{and} \quad y_1 = \sum_{m=0}^{\infty} c_m x^{m+r_2}.$$

Find c_m 's.

a. Let $y = \sum_{m=0}^{\infty} c_m x^{m+r}$. Then

$$y' = \sum_{m=0}^{\infty} c_m(m+r)x^{m+r-1}, \quad y'' = \sum_{m=0}^{\infty} c_m(m+r)(m+r-1)x^{m+r-2}.$$

b. Substitute y , y' and y'' into the differential equation:

$$\begin{aligned} & 3x \sum_{m=0}^{\infty} c_m(m+r)(m+r-1)x^{m+r-2} + \sum_{m=0}^{\infty} c_m(m+r)x^{m+r-1} - \sum_{m=0}^{\infty} c_m x^{m+r} \\ &= \sum_{m=0}^{\infty} 3c_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{\infty} c_m(m+r)x^{m+r-1} - \sum_{m=0}^{\infty} c_m x^{m+r} = 0 \end{aligned}$$

Replace m by k in the 1st and 2nd summation and replace m by $k-1$ in the 3rd summation. For the 3rd summation:

$$m = k-1, \quad k = m+1, \quad \text{when } m=0, \quad k=1.$$

$$\begin{aligned} & \sum_{k=0}^{\infty} 3c_k(k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k(k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1}x^{k+r-1} \\ &= \sum_{k=0}^{\infty} 3c_k(k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k(k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1}x^{k+r-1} \\ &= [3c_0(r)(r-1) + c_0(r)]x^{r-1} + \sum_{k=1}^{\infty} [3c_k(k+r)(k+r-1) + c_k(k+r) - c_{k-1}]x^{k+r-1} \\ &= c_0(r)(3(r-1) + 1)x^{r-1} + \sum_{k=1}^{\infty} [c_k(k+r)(3(k+r-1) + 1) - c_{k-1}]x^{k+r-1} \\ &= c_0(r)(3r-2)x^{r-1} + \sum_{k=1}^{\infty} [c_k(k+r)(3k+3r-2) - c_{k-1}]x^{k+r-1} = 0 \end{aligned}$$

c. The coefficient of x^{r-1} should be 0 :

$$c_0 r(3r-2) = 0.$$

Since $c_0 \neq 0$, $r = 0$ or $r = \frac{2}{3}$. (r 's should be the same as the roots of the corresponding indicial equation.) Coefficients of x^{k+r-1} should be 0 :

$$c_k(k+r)(3k+3r-2) - c_{k-1} = 0$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1}, \quad k = 1, 2, \dots$$

$$r = 0, \quad c_k = \frac{1}{(k)(3k-2)} c_{k-1}, \quad k = 1, 2, \dots$$

$$r = \frac{2}{3}, \quad c_k = \frac{1}{(k + \frac{2}{3})(3k)} c_{k-1} = \frac{1}{(k)(3k+2)} c_{k-1}, \quad k = 1, 2, \dots$$

Derive recurrence relations of c_k for $r = 0$ and $r = \frac{2}{3}$:

i. $r = 0$,

$k = 1$	$c_1 = \frac{1}{(1)(1)}c_0 = c_0$
$k = 2$	$c_2 = \frac{1}{(2)(4)}c_1 = \frac{1}{(2!)(4)(1)}c_0 = \frac{1}{8}c_0$
$k = 3$	$c_3 = \frac{1}{(3)(7)}c_2 = \frac{1}{(3!)(7)(4)(1)}c_0 = \frac{1}{168}c_0$
\vdots	\vdots
$k = m$	$c_m = \frac{1}{(m!)(3m-2)(3m-5)\dots(7)(4)(1)}c_0$

ii. $r = \frac{2}{3}$,

$k = 1$	$c_1 = \frac{1}{(1)(5)}c_0 = \frac{1}{5}c_0$
$k = 2$	$c_2 = \frac{1}{(2)(8)}c_1 = \frac{1}{(2!)(8)(5)}c_0 = \frac{1}{80}c_0$
$k = 3$	$c_3 = \frac{1}{(3)(11)}c_2 = \frac{1}{(3!)(11)(8)(5)}c_0 = \frac{1}{2640}c_0$
\vdots	\vdots
$k = m$	$c_m = \frac{1}{(m!)(3m+2)(3m+1)\dots(11)(8)(5)}c_0$

d.

$$\begin{aligned}
 y &= \sum_{m=0}^{\infty} c_m x^{m+r_1} + \sum_{m=0}^{\infty} c_m x^{m+r_2} \\
 &= c_{01} \left(1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \dots \right) + c_{02}x^{2/3} \left(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \frac{1}{2640}x^3 + \dots \right) \\
 &= c_{01} \left(1 + \sum_{m=1}^{\infty} \frac{1}{(m!)(3m-2)(3m-5)\dots(7)(4)(1)} x^m \right) \\
 &\quad + c_{02}x^{2/3} \left(1 + \sum_{m=1}^{\infty} \frac{1}{(m!)(3m+2)(3m+1)\dots(11)(8)(5)} x^m \right)
 \end{aligned}$$

Example Solve $xy'' + y = 0$.

$a_2(x) = x = 0$ when $x = 0$. Check the differential equation: $y'' + \frac{1}{x}y = 0$

$$xP(x) = x(0) = 0 \quad (a_0 = 0)$$

$$x^2Q(x) = x^2\left(\frac{1}{x}\right) = x \quad (b_0 = 0)$$

both are analytic at $x = 0$. So, $x = 0$ is a regular singular point of the equation. The indicial equation is

$$(r)(r-1) + a_0r + b_0 = (r)(r-1) = 0$$

and its roots are:

$$r = 0 \quad \text{and} \quad r = 1.$$

Since $r_2 - r_1 = 1$ is a positive integer, there may be only one solution in the form of a power series centered at 0 : $y = \sum_{m=0}^{\infty} c_m x^{m+r}$. Find c_m 's.

a. Let $y = \sum_{m=0}^{\infty} c_m x^{m+r}$. Then

$$y' = \sum_{m=0}^{\infty} c_m(m+r)x^{m+r-1}, \quad y'' = \sum_{m=0}^{\infty} c_m(m+r)(m+r-1)x^{m+r-2}.$$

b. Substitute y , and y'' into the differential equation:

$$\begin{aligned} & x \sum_{m=0}^{\infty} c_m(m+r)(m+r-1)x^{m+r-2} + \sum_{m=0}^{\infty} c_m x^{m+r} \\ &= \sum_{m=0}^{\infty} c_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{\infty} c_m x^{m+r} = 0 \end{aligned}$$

Replace m by k in the 1st and replace m by $k-1$ in the 2nd summation:

$$m = k - 1, \quad k = m + 1, \quad \text{when } m = 0, \quad k = 1.$$

$$\begin{aligned} & \sum_{k=0}^{\infty} c_k(k+r)(k+r-1)x^{k+r-1} + \sum_{k=1}^{\infty} c_{k-1}x^{k+r-1} \\ &= c_0(r)(r-1)x^{r-1} + \sum_{k=1}^{\infty} [c_k(k+r)(k+r-1) + c_{k-1}]x^{k+r-1} = 0 \end{aligned}$$

c. The coefficient of x^{r-1} is 0 :

$$c_0 r(r-1) = 0.$$

Since $c_0 \neq 0$, $r = 0$ or $r = 1$ (Again, r 's are the same as the solutions of the indicial equation.)

Coefficients of x^{k+r-1} are 0 :

$$c_k(k+r)(k+r-1) + c_{k-1} = 0 \quad \Rightarrow \quad c_k = -\frac{1}{(k+r)(k+r-1)}c_{k-1}, \quad k = 1, 2, \dots$$

$$\text{For } r = 0, \quad c_k = -\frac{1}{(k)(k-1)}c_{k-1}, \quad k = 1, 2, 3, \dots \quad (\text{it is not defined for } k = 1).$$

$$\text{For } r = 1, \quad c_k = \frac{1}{(k+1)(k)}c_{k-1}, \quad k = 1, 2, \dots$$

i. $r = 0$. The recurrence relation for c_k is not defined. What does this mean? Observe that when $k = 1$, $c_k(k+r)(k+r-1) + c_{k-1} = c_1(1)(0) + c_0 = c_0 \neq 0$.

So, $y = \sum_{m=0}^{\infty} c_m x^m$ is not a solution of the equation.

ii. $r = 1$,

$k = 1$	$c_1 = -\frac{1}{(1)(2)}c_0 = -\frac{1}{2}c_0$
$k = 2$	$c_2 = -\frac{1}{(2)(3)}c_1 = \frac{(-1)^2}{(2!)(3!)}c_0 = \frac{1}{12}c_0$
$k = 3$	$c_3 = -\frac{1}{(3)(4)}c_2 = \frac{(-1)^3}{(3!)(4!)}c_0 = -\frac{1}{144}c_0$
\vdots	\vdots
$k = m$	$c_m = \frac{(-1)^m}{(m!)(m+1)!}c_0$

d.

$$\begin{aligned} y_1 &= c_0 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \dots \right) \\ &= c_0 x \left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)(m+1)!} x^m \right) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)(m+1)!} x^{m+1} \end{aligned}$$

$$y_2 = y_1 \int \frac{1}{y_1^2} \left(e^{-\int P(x) dx} \right) dx.$$

$$P(x) = 0, \quad \int P(x) dx = C, \quad e^{-\int P(x) dx} = e^{-C} = C$$

$$\int \frac{1}{y_1^2} dx = \int \frac{1}{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots\right)^2} dx$$

$$\frac{1}{x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4} \approx x^{-1} + \frac{1}{2} + \frac{1}{6}x + \frac{7}{144}x^2 + \frac{1}{72}x^3 + \frac{7}{1728}x^4$$

$$\begin{aligned} \frac{1}{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4\right)^2} &= \left(x^{-1} + \frac{1}{2} + \frac{1}{6}x + \frac{7}{144}x^2 + \frac{1}{72}x^3 + \frac{7}{1728}x^4\right)^2 \\ &= \frac{229}{20736}x^4 + \frac{11}{288}x^3 + \frac{5}{48}x^2 + \frac{19}{72}x + \frac{7}{12} + \frac{1}{x} + \frac{1}{x^2} \end{aligned}$$

$$\begin{aligned} &\int \frac{1}{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4\right)^2} dx \\ &= \int \left(x^{-2} + x^{-1} + \frac{7}{12} + \frac{19}{72}x + \frac{5}{48}x^2 + \frac{11}{288}x^3 + \frac{229}{20736}x^4\right) dx \\ &= -\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \frac{5}{144}x^3 + \frac{11}{1152}x^4 + \frac{229}{103680}x^5 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 \left(-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \frac{5}{144}x^3 + \frac{11}{1152}x^4 + \dots\right) \\ &= y_1 \ln x + y_1 \left(-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \frac{5}{144}x^3 + \frac{11}{1152}x^4 + \dots\right) \\ &= y_1 \ln x + \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots\right) \left(-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \frac{5}{144}x^3 + \frac{11}{1152}x^4 + \dots\right) \\ &= y_1 \ln x + \\ &\quad \left(-1 + \frac{1}{2}x^2 - \frac{11}{72}x^3 + \frac{5}{288}x^4 - \frac{1}{1152}x^5 + \frac{1}{2}x - \frac{29}{10368}x^6 + \frac{23}{41472}x^7 - \frac{11}{165888}x^8 + \dots\right) \end{aligned}$$

Observe that y_2 is of the form

$$y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m.$$

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 \\ &= C_1 \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots\right) + C_2 \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots\right) \ln x \\ &\quad + \left(-1 + \frac{1}{2}x^2 - \frac{11}{72}x^3 + \frac{5}{288}x^4 - \frac{1}{1152}x^5 + \frac{1}{2}x - \frac{29}{10368}x^6 + \frac{23}{41472}x^7 - \frac{11}{165888}x^8 + \dots\right) \end{aligned}$$

e. Use a **TI-89 calculator** to find an approximation of y_2 :

$$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots \approx x - \frac{1}{2}x^2, \quad P(x) = 0$$

i. $y_1 \approx x - \frac{1}{2}x^2 = x\left(1 - \frac{1}{2}x\right)$
 $\frac{1}{y_1} = \frac{1}{x\left(1 - \frac{1}{2}x\right)} = \frac{1}{x} - \frac{1}{x-2}$, **F2, 3:expand(: expand(1/(x*(1-x/2)),x)**

F3, 9:taylor(: taylor(-1/(x-2),x,3)

Answer: $\frac{x^3}{16} + \frac{x^2}{8} + \frac{x}{4} + \frac{1}{2}$, $\frac{1}{y_1} = \frac{1}{x} + \frac{x^3}{16} + \frac{x^2}{8} + \frac{x}{4} + \frac{1}{2}$

ii. **F2, 3:expand(: expand((1/x+x^3/16+x^2/8+x/4+1/2)^2,x)**

answer: $\frac{x^6}{256} + \frac{x^5}{64} + \frac{3x^4}{64} + \frac{x^3}{8} + \frac{5x^2}{16} + \frac{x}{2} + \frac{1}{x} + \frac{1}{x^2} + \frac{3}{4}$

Now we have $\frac{1}{y_1^2} \approx \frac{x^3}{8} + \frac{5x^2}{16} + \frac{x}{2} + \frac{1}{x} + \frac{1}{x^2} + \frac{3}{4}$

iii. $\int P(x)dx = \int 0dx = C, \quad e^{-\int P(x)dx} = e^C = C$

To evaluate $\int \frac{1}{y_1^2} e^{-\int P(x)dx} dx = \int \left(\frac{x^3}{8} + \frac{5x^2}{16} + \frac{x}{2} + \frac{1}{x} + \frac{1}{x^2} + \frac{3}{4} \right) dx$

F3, 2: $\int \left(\frac{x^3}{8} + \frac{5x^2}{16} + \frac{x}{2} + \frac{1}{x} + \frac{1}{x^2} + \frac{3}{4} \right) dx$

answer: $\ln|x| + \frac{x^4}{32} + \frac{5x^3}{40} + \frac{x^2}{4} + \frac{3x}{4} - \frac{1}{x}$

iv. To evaluate $y_2 = \left(x \left(1 - \frac{1}{2}x \right) \right) \left(\ln|x| + \frac{x^4}{32} + \frac{5x^3}{40} + \frac{x^2}{4} + \frac{3x}{4} - \frac{1}{x} \right)$