Method of Variation of Parameters for Nonhomogeneous Linear Differential Equations - (3.5)

Consider the general solution of an nth-order nonhomogeneous linear differential equation:

$$L(y) = f(x) \text{ where } L(y) = y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y.$$

Suppose that the general solution $y_c = C_1y_1 + \ldots + C_ny_n$ of the corresponding homogeneous differential equation $L(y) = 0$ is given. The Method of Variation of Parameters solves y_p as follows. Let

$$y_p = u_1y_1 + u_2y_2 + \ldots + u_n\, y_n.$$

Find $u_1, \ldots, u_n$ so that y_p is a solution of $L(y) = f(x)$.

1. First let us consider the second order differential equation of the form: $y'' + P(x)y' + Q(x)y = f(x)$.
Suppose we know $y_c = c_1y_1 + c_2y_2$. Let $y_p = u_1y_1 + u_2y_2$ where u_1 and u_2 are functions of x. Two steps to solve u_1 and u_2 :

Step I Solve u_1' and u_2' from the system of two linear equations:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned} \quad \text{or} \quad \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Step II $u_i = \int u_i' dx$, for $i = 1, 2$.

The derivation:

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Set

$$u_1'y_1 + u_2'y_2 = 0, \quad (*)$$

then

$$y_p' = u_1y_1' + u_2y_2'.$$

Compute

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2 + u_2y_2''.$$

Substitute y_p, y_p' and y_p'' into the differential equation: $y'' + P(x)y' + Q(x)y = f(x)$

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) = f(x)$$

Rewrite the equation as

$$u_1(y_1'' + P(x)y_1' + Q(x)y_1) + u_2(y_2'' + P(x)y_2' + Q(x)y_2) + u_1'y_1' + u_2'y_2' = f(x).$$

Because y_1 and y_2 are solutions of

$$y'' + P(x)y' + Q(x)y = f(x),$$

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \text{ and } y_2'' + P(x)y_2' + Q(x)y_2 = 0.$$

So, the differential equation can be simplified as

$$u_1'y_1' + u_2'y_2' = f(x). \quad (**)$$

Equations $(*)$ and $(**)$ are two equations for solving u_1' and u_2'.

2. The nth order differential equation: $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x)$.
Suppose that we know $y_c = C_1y_1 + \ldots + C_ny_n$. Let $y_p = u_1y_1 + u_2y_2 + \ldots + u_ny_n$ where $u_1, \ldots, u_n$ are functions of x. Two steps to solve $u_1, \ldots, u_n$:

Step I Solve $u_1', \ldots u_n'$ from the system of n linear equations:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix}$$

Step II $u_i = \int u_i' dx$, for $i = 1, 2, \ldots, n$.

The derivation is similar to the one for the second order differential equation.

Example *Solve the following equations.*

$$(a)\ y'' + 9y = csc\,3x \quad (b)\ y'' + \pi^2 y = sin(\pi x) \quad (c)\ y''' + 3y'' + 3y' + y = 2e^x + x$$

(***a***) $y'' + 9y = \csc 3x$

a. Solve y_c from $y'' + 9y = 0$.

$P(m) = m^2 + 9 = 0,\ m = \pm i3, \quad y_1 = \cos(3x), \quad y_2 = \sin(3x)$

$y_c = c_1 \cos(3x) + c_2 \sin(3x)$

b. Let $y_p = u_1 \cos(3x) + u_2 \sin(3x)$.

Solve u_1' and u_2' from the system:

$$\begin{pmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \csc(3x) \end{pmatrix}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \csc(3x) \end{pmatrix}$$

$$= \frac{1}{3\cos^2(3x) + 3\sin^2(3x)} \begin{pmatrix} 3\cos(3x) & -\sin(3x) \\ 3\sin(3x) & \cos(3x) \end{pmatrix} \begin{pmatrix} 0 \\ \csc(3x) \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} -\sin(3x)\csc(3x) \\ \cos(3x)\csc(3x) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3}\frac{\cos(3x)}{\sin(3x)} \end{pmatrix}$$

Solve u_1 and u_2 :

$$u_1 = \int u_1' dx = \int -\frac{1}{3} dx = -\frac{1}{3}x$$

$$u_2 = \int u_2' dx = \int \frac{1}{3}\frac{\cos(3x)}{\sin(3x)} dx = \frac{1}{9}\ln|\sin(3x)|$$

$$y_p = -\frac{1}{3}x\cos(3x) + \frac{1}{9}\ln|\sin(3x)|\sin(3x)$$

and the general solution of the differential equation is

$$y = c_1\cos(3x) + c_2\sin(3x) + -\frac{1}{3}x\cos(3x) + \frac{1}{9}\ln|\sin(3x)|\sin(3x).$$

(***b***) $y'' + \pi^2 y' = \sin(\pi x),\ y(0) = 1,\ y'(0) = -1.$

a. Solve y_c from $y'' + \pi^2 y = 0$.

$P(m) = m^2 + \pi^2 = 0,\ m = \pm i\pi, \quad y_1 = \cos(\pi x),\ y_2 = \sin(\pi x).$

$y_c = c_1\cos(\pi x) + c_2\sin(\pi x)$

b. Let $y_p = u_1\cos(\pi x) + u_2\sin(\pi x)$.

i. Solve u_1' and u_2' from the system:

$$\begin{pmatrix} \cos(\pi x) & \sin(\pi x) \\ -\pi\sin(\pi x) & \pi\cos(\pi x) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(\pi x) \end{pmatrix}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos(\pi x) & \sin(\pi x) \\ -\pi\sin(\pi x) & \pi\cos(\pi x) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \sin(\pi x) \end{pmatrix}$$

$$= \frac{1}{\pi\cos^2(\pi x) + \pi\sin^2(\pi x)} \begin{pmatrix} \pi\cos(\pi x) & -\sin(\pi x) \\ \pi\sin(\pi x) & \cos(\pi x) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\pi x) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\pi}\sin^2\pi x \\ \frac{1}{\pi}\cos\pi x\sin\pi x \end{pmatrix}$$

ii. Solve u_1 and u_2 :

$$u_1 = \int -\frac{1}{\pi}\sin^2\pi x dx = -\frac{1}{\pi}\int \frac{1}{2}(1-\cos(2\pi x))dx = -\frac{1}{2\pi}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)$$

$$u_2 = \int \frac{1}{\pi}\cos\pi x\sin\pi x dx = \frac{1}{\pi}\left(\frac{1}{\pi}\sin^2(\pi x)\right) = \frac{1}{\pi^2}\sin^2(\pi x)$$

$$y_p = -\frac{1}{2\pi}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)\cos(\pi x) + \frac{1}{\pi^2}\sin^2(\pi x)\sin(\pi x)$$

and the general solution of the differential equation is:

$$y = c_1\cos(\pi x) + c_2\sin(\pi x) - \frac{1}{2\pi}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)\cos(\pi x) + \frac{1}{\pi^2}\sin^3(\pi x)$$

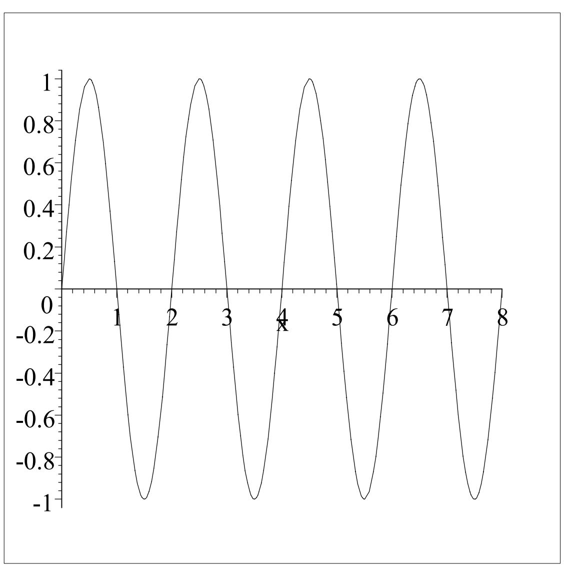
c. Solve c_1 and c_2 using the initial value conditions: $y(0) = 1,\ \ y'(0) = -1$

$$y' = -\pi c_1\sin(\pi x) + \pi c_2\cos(\pi x) - \frac{1}{2\pi}(1-\cos(2\pi x))\cos(\pi x) + \frac{1}{2}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)\sin(\pi x)$$
$$+ \frac{3}{\pi}\sin^2(\pi x)\cos(\pi x)$$

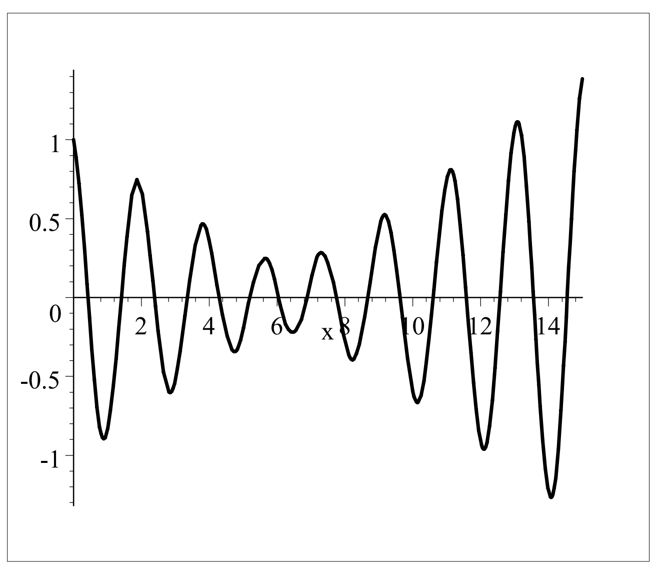
$$y(0) = c_1 = 1, \quad y'(0) = \pi c_2 = -1,\ c_2 = -\frac{1}{\pi}$$

The solution of the initial value problem is:

$$y = \cos(\pi x) - \frac{1}{\pi}\sin(\pi x) - \frac{1}{2\pi}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)\cos(\pi x) + \frac{1}{\pi^2}\sin^3(\pi x)$$



$f(x) = \sin(\pi x)$



y

(c) $y''' - 3y'' + 3y' - y = 2e^x + x$

a. Solve y_c from $y''' - 3y'' + 3y' - y = 0$.

$$P(m) = m^3 - 3m^2 + 3m - 1 = m^3 - 1 - 3m(m-1) = (m-1)(m^2 + m + 1) - 3m(m-1)$$
$$= (m-1)(m^2 + m + 1 - 3m) = (m-1)(m^2 - 2m + 1) = (m-1)^3 = 0,\ m = 1,1,1$$
$$y_1 = e^x,\ y_2 = xe^x,\ y_3 = x^2e^x,\quad y_c = c_1e^x + c_2xe^x + c_3x^2e^x$$

b. Let $y_p = u_1e^x + u_2xe^x + u_3x^2e^x$.

i. Solve $u_1',\ u_2'$ and u_3' :

$$y_2 = xe^x,\ y_2' = e^x + xe^x = (1+x)e^x,\ y_2'' = e^x + (1+x)e^x = (2+x)e^x$$
$$y_3 = x^2e^x,\ y_3' = 2xe^x + x^2e^x = (2x + x^2)e^x,\ y_3'' = (2+2x)e^x + (2x + x^2)e^x = (2 + 4x + x^2)e^x$$
$$\begin{pmatrix} e^x & xe^x & x^2e^x \\ e^x & (1+x)e^x & (2x+x^2)e^x \\ e^x & (2+x)e^x & (2+4x+x^2)e^x \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2e^x + x \end{pmatrix}$$

Simplify the system by multiplying e^{-x} both sides:

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ 2+xe^{-x} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2+xe^{-x} \end{pmatrix}$$

$$\left(A \mid b\right) = \left(\begin{array}{ccc|c} 1 & x & x^2 & 0 \\ 1 & 1+x & 2x+x^2 & 0 \\ 1 & 2+x & 2+4x+x^2 & 2+xe^{-x} \end{array}\right) \begin{array}{c} -R_1 + R_2 \to R_2 \\ \Rightarrow \\ -R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 2 & 2+4x & 2+xe^{-x} \end{array}\right) \begin{array}{c} -2R_2 + R_3 \to R_3 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 0 & 2 & 2+xe^{-x} \end{array}\right)$$

$$u_3' = \frac{1}{2}(2 + xe^{-x}),$$
$$u_2' = -2x\left(\frac{1}{2}(2 + xe^{-x})\right) = -x(2 + xe^{-x})$$
$$u_1' = -x(-x(2 + xe^{-x})) - x^2\left(\frac{1}{2}(2 + xe^{-x})\right) = 2x^2 + x^3e^{-x} - x^2 - \frac{1}{2}x^3e^{-x} = x^2 + \frac{1}{2}x^3e^{-x}$$

ii. Solve $u_1,\ u_2$ and u_3 :

$$u_1 = \int\left(x^2 + \frac{1}{2}x^3e^{-x}\right)dx = \frac{1}{3}x^3 - \frac{1}{2}x^3e^{-x} - \frac{3}{2}x^2e^{-x} - 3xe^{-x} - 3e^{-x}$$
$$u_2 = \int -x(2 + xe^{-x})dx = -x^2 + x^2e^{-x} + 2xe^{-x} + 2e^{-x}$$
$$u_3 = \int \frac{1}{2}(2 + xe^{-x})dx = x - \frac{1}{2}xe^{-x} - \frac{1}{2}e^{-x}$$

$$y_p = \left(\frac{1}{3}x^3 - \frac{1}{2}x^3e^{-x} - \frac{3}{2}x^2e^{-x} - 3xe^{-x} - 3e^{-x}\right)e^x + (-x^2 + x^2e^{-x} + 2xe^{-x} + 2e^{-x})xe^x$$
$$+ \left(x - \frac{1}{2}xe^{-x} - \frac{1}{2}e^{-x}\right)x^2e^x$$
$$= \frac{1}{3}(x^3 - 3xe^{-x} - 9e^{-x})e^x$$

The general solution of the differential equation is:

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{3}(x^3 - 3xe^{-x} - 9e^{-x})e^x$$