Homogeneous Linear Differential Equations with Constant Coefficients - (3.3)

Consider an nth-order linear differential equation of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-2)} + \ldots + a_1 y' + a_0 y = 0.$$

Let $y = e^{mx}$. Observe that

$$y' = me^{mx},\ y'' = m^2 e^{mx},\ \ldots,\ y^{(n)} = m^n e^{mx}.$$

$$a_n y^{(n)} + a_{n-1} y^{(n-2)} + \ldots + a_1 y' + a_0 y = a_n m^n e^{mx} + \ldots + a_1 m e^{mx} + a_0 e^{mx}$$

$$= e^{mx}(a_n m^n + \ldots + a_1 m + a_0) = 0 \Rightarrow$$

$$a_n m^n + \ldots + a_1 m + a_0 = 0.$$

$y = e^{ax}$ is a solution of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-2)} + \ldots + a_1 y' + a_0 y = 0$$

if and only if $m = a$ is a solution of the equation

$$a_n m^n + \ldots + a_1 m + a_0 = 0.$$

Let $P(m) = a_n m^n + \ldots + a_1 m + a_0$. $P(m)$ is called the **characteristic polynomial** of the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-2)} + \ldots + a_1 y' + a_0 y = 0.$$

Know that the equation $P(m) = m$ has n solutions (real or complex) including the multiples. Let $m_1, \ldots, m_n$ be solutions of $P(m) = 0$. Let $L(y) = a_n y^{(n)} + a_{n-1} y^{(n-2)} + \ldots + a_1 y' + a_0 y$. Recall the general solution of $L(y) = 0$:

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n$$

where $y_1, \ldots, y_n$ are solutions of $L(y) = 0$ and are linearly independent. How do $y_1, \ldots, y_n$ relate to $m_1, \ldots, m_n$?

1. If m_i are simple real solutions of $P(m) = 0$, then $y_i = e^{m_i x}$ are solutions of the differential equation $L(y) = 0$.
2. If m_i is a real solution of $P(m) = 0$ with multiplicity k, then $y_1 = e^{m_i x}$, $y_2 = xe^{m_i x}$, $\ldots$, $y_k = x^{k-1} e^{m_i x}$ are solutions of the differential equation $L(y) = 0$ and they are linearly independent.
3. If $m_1 = a + ib$ and $m_2 = a - ib$ are simple complex solutions of $P(m) = 0$, then
$$y_1 = e^{ax}\cos(bx),\ y_2 = e^{ax}\sin(bx)$$
are solutions of $L(y) = 0$ and they are linearly independent.
4. If $m_1 = a + ib$ and $m_2 = a - ib$ are complex solutions of $P(m) = 0$ with multiplicity k, then
$$y_1 = e^{ax}\cos(bx),\ y_2 = e^{ax}\sin(bx),\ y_3 = xe^{ax}\cos(bx),\ y_4 = xe^{ax}\sin(bx)$$
$$\ldots y_{2k-1} = x^{k-1}e^{ax}\cos(bx),\ y_{2k} = x^{k-1}e^{ax}\sin(bx)$$
are solutions of $L(y) = 0$ and they are linearly independent.

Example *Let $P(m) = m^2(m-1)^2(m+2)(m^2+3)(m^2+m+1)$ be the characteristic polynomial of a linear differential equation $L(y) = 0$. What is the order the differential equation? Find the general solution of $L(y) = 0$.*

$P(m)$ is a polynomial of degree 9, so the order of differential equation is 9. Solve $P(m) = 0$.

$$\begin{aligned}
&m^2 = 0,\ m = 0, 0 && y_1 = 1,\ y_2 = x\\
&(m-1)^2 = 0,\ m = 1, 1 && y_3 = e^x,\ y_4 = xe^x\\
&m + 2 = 0,\ m = -2 && y_5 = e^{-2x}\\
&m^2 + 3 = 0,\ m = i\sqrt{3},\ m = -i\sqrt{3} \quad \Rightarrow && y_6 = \cos\left(\sqrt{3}x\right),\ y_7 = \sin\left(\sqrt{3}x\right)\\
&m^2 + m + 1 = 0,\ m = \tfrac{-1\pm\sqrt{1-4}}{2} && y_8 = e^{-x/2}\cos\left(\tfrac{\sqrt{3}}{2}x\right),\\
&\quad m = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2},\ m = -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} && y_9 = e^{-x/2}\sin\left(\tfrac{\sqrt{3}}{2}x\right)
\end{aligned}$$

The general solution:

$$y = C_1 + C_2x + C_3e^x + C_4xe^x + C_5e^{-2x} + C_6\cos\left(\sqrt{3}\,x\right) + C_7\sin\left(\sqrt{3}\,x\right)$$
$$+ C_8e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_9e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Example *Find the general solution of the differential equation.*

$a.\ y'' - y' - 2y = 0 \quad b.\ y'' + y' + 2y = 0 \quad c.\ y'' + 4y' + 4y = 0$

$d.\ y^{(4)} - y' = 0 \quad e.\ y^{(4)} - 2y'' = 0 \quad f.\ y^{(4)} + 4y'' + 4y = 0 \quad g.\ y^{(4)} - y = 0$

a. $P(m) = m^2 - m - 2 = (m-2)(m+1) = 0,\ m = 2$, and $m = -1$.
$y_1 = e^{2x},\ y_2 = e^{-x}$, and the general solution

$$y = C_1e^{2x} + C_2e^{-x}.$$

b. $P(m) = m^2 + m + 2 = 0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(2)}}{2} = \frac{-1 \pm i\sqrt{7}}{2},\ m = -\frac{1}{2} + i\frac{\sqrt{7}}{2},\ m = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$$

$$y_1 = e^{-x/2}\cos\left(\frac{\sqrt{7}}{2}\right),\ y_2 = e^{-x/2}\sin\left(\frac{\sqrt{7}}{2}\right),\ \text{and } y = C_1e^{-x/2}\cos\left(\frac{\sqrt{7}}{2}\right) + C_2e^{-x/2}\sin\left(\frac{\sqrt{7}}{2}\right)$$

c. $P(m) = m^2 + 4m + 4 = (m+2)^2 = 0,\ m = -2, -2.$
$y_1 = e^{-2x},\ y_2 = xe^{-2x}$ and

$$y = C_1e^{-2x} + C_2xe^{-2x}$$

d. $P(m) = m^4 - m = m(m^3 - 1) = m(m-1)(m^2 + m + 1) = 0,$
$m = 1,\ m = 1,\ m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
$y_1 = e^x,\ y_2 = xe^x,\ y_3 = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ y_4 = e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$ and

$$y = C_1e^x + C_2xe^x + C_3e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_4e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

e. $P(m) = m^4 - 2m^2 = m^2(m^2 - 2) = 0,\ m = 0, 0,\ m = \pm\sqrt{2}$
$y_1 = 1,\ y_2 = x,\ y_3 = e^{\sqrt{2}x},\ y_4 = e^{-\sqrt{2}x}$ and

$$y = C_1 + C_2x + C_3e^{\sqrt{2}x} + C_4e^{-\sqrt{2}x}.$$

f. $P(m) = m^4 + 4m^2 + 4 = (m^2 + 2)^2 = 0,\ m = \pm i\sqrt{2},\ m = \pm i\sqrt{2}$
$y_1 = \cos\left(\sqrt{2}\,x\right),\ y_2 = \sin\left(\sqrt{2}\,x\right),\ y_3 = x\cos\left(\sqrt{2}\,x\right),\ \ y_4 = x\sin\left(\sqrt{2}\,x\right)$ and

$$y = C_1\cos\left(\sqrt{2}\,x\right) + C_2\sin\left(\sqrt{2}\,x\right) + C_3x\cos\left(\sqrt{2}\,x\right) + C_4x\sin\left(\sqrt{2}\,x\right)$$

g. $P(m) = m^4 - 1 = (m^2 - 1)(m^2 + 1) = 0,\ m = \pm 1,\ m = \pm i$
$y_1 = e^x,\ y_2 = e^{-x},\ y_3 = \cos x,\ y_4 = \sin x$ and

$$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x.$$