

Reduction of Order - (3.2)

1. Consider the 2nd-order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

The general solution of the differential equation is $y = C_1y_1 + C_2y_2$ where $\{y_1, y_2\}$ is a fundamental set of solutions, i.e.

- a. y_1 , and y_2 are solutions of the differential equation; and
- b. y_1 , and y_2 are linearly independent.

If we know y_1 , can we find y_2 ? The answer is yes. How? We let $y_2 = u y_1$ where u is a function of x . If we substitute y_2 of this form into the differential equation, we will get a first-order differential equation in u and it can be solved by using a method Chapter 2. Here are the details.

Consider the 2nd-order homogeneous linear differential equation in standard form

$$y'' + P(x)y' + Q(x)y = 0$$

Let $y_2 = y_1u$. Then

$$y_2' = y_1'u + y_1u' \text{ and}$$

$$y_2'' = y_1''u + y_1'u' + y_1'u' + y_1u'' = y_1''u + 2y_1'u' + y_1u''.$$

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= y_1''u + 2y_1'u' + y_1u'' + P(x)(y_1'u + y_1u') + Q(x)y_1u \\ &= (y_1'' + P(x)y_1' + Q(x)y_1)u + 2y_1'u' + y_1u'' + P(x)y_1u' \\ &= 0 + y_1u'' + (2y_1' + P(x)y_1)u' = 0 \end{aligned}$$

$$u'' + \left(\frac{2y_1'}{y_1} + P(x) \right) u' = 0, \text{ let } z = u'. \text{ Then } z' = u''$$

$$z' + \left(\frac{2y_1'}{y_1} + P(x) \right) z = 0 \quad - \quad \begin{array}{l} \text{1st order linear differential equation} \\ \text{separable} \end{array}$$

Solve the differential equation in z and then

$$u = \int z dx \text{ and } y_2 = y_1u$$

Given $y'' + P(x)y' + Q(x)y = 0$ and a solution y_1 , two steps to solve $y_2 = u y_1$:

- a. Solve the first order linear (separable) differential equation in z :

$$z' + \left(\frac{2y_1'}{y_1} + P(x) \right) z = 0$$

- b. Compute $u = \int z dx$ and let $y_2 = u y_1$.

Example Show that $y_1 = x^2 \cos(\ln x)$ is a solution of $x^2y'' - 3xy' + 5y = 0$. Find the general solution of the differential equation.

- a. Check y_1 is a solution of the differential equation (it satisfies the differential equation) :

$$y_1' = 2x \cos(\ln x) - x^2 \sin(\ln x) \frac{1}{x} = x(2 \cos(\ln x) - \sin(\ln x)),$$

$$y_1'' = 2 \cos(\ln x) - \sin(\ln x) + x \left(-2 \sin(\ln x) \frac{1}{x} - \cos(\ln x) \frac{1}{x} \right) = \cos(\ln x) - 3 \sin(\ln x)$$

$$\begin{aligned} x^2y'' - 3xy' + 5y &= x^2(\cos(\ln x) - 3 \sin(\ln x)) - 3x(x)(2 \cos(\ln x) - \sin(\ln x)) + 5x^2 \cos(\ln x) \\ &= (x^2 - 6x^2 + 5x^2) \cos(\ln x) + (-3x^2 + 3x^2) \sin(\ln x) = 0 \end{aligned}$$

So, y_1 is a solution.

- b. Let $y_2 = y_1u$. First rewrite the equation in standard form

$$y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0. \quad P(x) = -\frac{3}{x}, \quad Q(x) = \frac{5}{x^2}$$

i. Solve $z' + \left(\frac{2y_1'}{y_1} + P(x)\right)z = 0$ for z . $P(x) = -\frac{3}{x}$

$$\frac{2y_1'}{y_1} = \frac{2x(2\cos(\ln x) - \sin(\ln x))}{x^2 \cos(\ln x)} = \frac{2(2\cos(\ln x) - \sin(\ln x))}{x \cos(\ln x)}$$

$$\frac{2y_1'}{y_1} + P(x) = \frac{2(2\cos(\ln x) - \sin(\ln x))}{x \cos(\ln x)} - \frac{3}{x} = \frac{1}{x} - \frac{2}{x} \tan(\ln x)$$

$$z' + \left(\frac{1}{x} - \frac{2}{x} \tan(\ln x)\right)z = 0$$

$$\int \frac{1}{z} dz = \int \left(-\frac{1}{x} + \frac{2}{x} \tan(\ln x)\right) dx, \quad \ln|z| = -\ln|x| - 2\ln|\cos \ln(x)| + C$$

$$z = C \frac{1}{x} \frac{1}{\cos^2 \ln x} = \frac{C}{x} \sec^2 \ln x$$

ii. Solve for u : $u = \int C \frac{1}{x} \sec^2 \ln x dx = C \tan(\ln x)$ and $y_2 = u y_1$

$$y_2 = Cx^2 \cos(\ln x) \tan(\ln x) = Cx^2 \sin(\ln x)$$

$$y = C_1 x^2 \cos(\ln x) + C_2 x^2 \sin(\ln x)$$