Separable Variables - (2.2)

1. **Separable Equations:**

A first order differential equation is said to be separable if it is of the form

$$\frac{dy}{dx} = g(x)h(y).$$

2. **Method of Separation of Variables:**

Observe that a separable equation can be written as

$$\frac{1}{h(y)}dy = g(x)dx \quad \Leftrightarrow \quad \int \frac{1}{h(y)}dy = \int g(x)dx$$

If we know the antiderivatives of $\frac{1}{h(y)}$ and $g(x)$ are $H(y)$ and $G(x)$, then

$$H(y) = G(x) + C$$

is the solution of the differentiation equation $\frac{dy}{dx} = g(x)h(y)$.

Example *Solve* $(1+x)dy - ydx = 0$.

Determine first if the differential equation is separable. Since

$$(1+x)dy - ydx = 0 \quad \Leftrightarrow \quad \frac{1}{y}dy = \frac{1}{1+x}dx,$$

the differential equation is separable.

$$\int \frac{1}{y}dy = \int \frac{1}{1+x}dx, \qquad \ln|y| = \ln|1+x| + C$$

$$y = Ce^{\ln|1+x|} = C(1+x)$$

is the general solution.

Example *Solve the initial value problem:*

$$cos(x)(e^{2y} - y)\frac{dy}{dx} = e^y sin(2x), \quad y(0) = 0.$$

Determine first if the differential equation is separable. Since

$$\frac{e^{2y} - y}{e^y}dy = \frac{\sin(2x)}{\cos(x)}dx,$$

the differential equation is separable.

$$\int \frac{e^{2y} - y}{e^y}dy = \int (e^{2y-y} - ye^{-y})dy = \int (e^y - ye^{-y})dy$$

$$\int \frac{\sin(2x)}{\cos(x)}dx = \int \frac{2\sin(x)\cos(x)}{\cos(x)}dx = 2\int \sin(x)dx,$$

$$\int (e^y - ye^{-y})dy = e^y - ye^{-y} - e^{-y} + C_1, \quad 2\int \sin(x)dx = -2\cos(x) + C$$

Hence, $e^y - ye^{-y} - e^{-y} = -2\cos(x) + C$ is the general solution.

Example *Solve* $\frac{dy}{dx} = y^2 - 4$.

Determine first if the differential equation is separable. Since

$$\frac{1}{y^2 - 4}dy = dx \text{ if } y^2 - 4 \neq 0,$$

the differential equation is separable wherever $y^2 - 4 \neq 0$.

$$\int \frac{1}{y^2-4}dy = \int dx \iff \int \frac{dy}{(y+2)(y-2)} = \int dx, \quad \int dx = x + C$$

$$\int \frac{dy}{(y+2)(y-2)} = \frac{1}{4}\int\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dx = \frac{1}{4}(\ln|y-2| - \ln|y+2|) = \frac{1}{4}\ln\left(\frac{y-2}{y+2}\right)$$

$$\frac{1}{4}\ln\left(\frac{y-2}{y+2}\right) = x + C, \quad \ln\left(\frac{y-2}{y+2}\right) = 4(x+C)$$

$$\frac{y-2}{y+2} = e^{4(x+C)} = e^{4x+4C} = e^{4x}e^{4C} = Ce^{4x}$$

Solve y in terms of x :

$$\frac{y-2}{y+2} = \frac{y+2-2-2}{y+2} = 1 - \frac{4}{y+2} = Ce^{4x}, \quad \frac{4}{y+2} = 1 - Ce^{4x}, \quad y+2 = \frac{4}{1-Ce^{4x}},$$

$$y = \frac{4}{1-Ce^{4x}} - 2 = \frac{4-2+2Ce^{4x}}{1-Ce^{4x}} = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$$

is the general solution of the equation. If $y^2 - 4 = 0$, then $y = \pm 2$. These are solutions.

Example $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3x+x-3}$

Determine first if the differential equation is separable. Since

$$\frac{dy}{dx} = \frac{y(x+2)-(x+2)}{y(x-3)+(x-3)} = \frac{(y-1)(x+2)}{(y+1)(x-3)} \iff \frac{y+1}{y-1}dy = \frac{x+2}{x-3}dx,$$

the differential equation is separable.

$$\int \frac{y+1}{y-1}dy = \int \frac{x+2}{x-3}dx \quad \Rightarrow \quad \int \frac{y-1+1+1}{y-1}dy = \int \frac{x-3+3+2}{x-3}dx$$

$$\int\left(1 + \frac{2}{y-1}\right)dy = \int\left(1 + \frac{5}{x-3}\right)dx$$

$$\int\left(1 + \frac{2}{y-1}\right)dy = y + 2\ln(y-1) + C_1, \quad \int\left(1 + \frac{5}{x-3}\right)dx = x + 5\ln(x-3) + C_2$$

The general solution of the differential equation is:

$$y + 2\ln(y-1) = x + 5\ln(x-3) + C$$

Example $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Determine first if the differential equation is separable. Since

$$e^x y\frac{dy}{dx} = e^{-y}(1+e^{-2x}) \iff \frac{y}{e^{-y}}dy = \frac{(1+e^{-2x})}{e^{-x}}dx$$

the differential equation is separable.

$$ye^y dy = (e^x + e^{-2x}e^x)dx \iff ye^y dy = (e^x + e^{-x})dx$$

$$\int ye^y dy = ye^y - e^y + C_1, \text{ and } \int(e^x + e^{-x})dx = e^x - e^{-x} + C_2$$

The general solution of the differential equation is:

$$ye^y - e^y = e^x - e^{-x} + C$$