Tangent Vector and Curvature - (11.4)

1. **Tangent Vector**

Let C be the curve traced out by the vector-valued function $\vec{r}(t) =< f(t),\ g(t),\ h(t) >$. The vector

$$\vec{T}(t) = \frac{1}{||\vec{r}\,'(t)||}\vec{r}\,'(t)$$

is the unit tangent vector to the curve C. Observe that

$$||\vec{r}\,'(t)|| = \sqrt{\vec{r}\,'(t) \cdot \vec{r}\,'(t)} = \sqrt{[f\,'(t)]^2 + [\,g'(t)\,]^2 + [h'(t)\,]^2}\,.$$

We know the arc length of C for $a \le t \le b$ is

$$L = \int_a^b \sqrt{[f\,'(t)]^2 + [\,g'(t)\,]^2 + [h'(t)\,]^2}\ dt = \int_a^b ||\vec{r}\,'(t)||dt$$

Example *Let $\vec{r}(t) =< 4\cos(t),\ \sin(t),\ t >$. Find the unit vector to the curve traced out by $\vec{r}(t)$. Sketch the unit tangent vectors at the points respectively when $t = 0,\ t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$. Evaluate the length of arc of the curve for $0 \le t \le \frac{\pi}{2}$.*

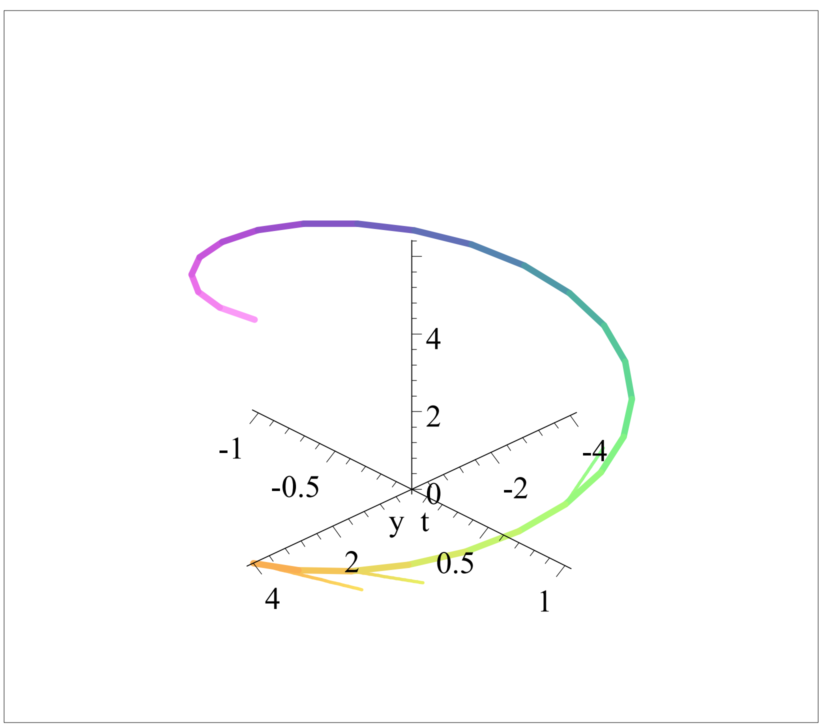
$\vec{r}\,'(t) =< -4\sin(t),\ \cos(t),\ 1 >$. $||\vec{r}(t)|| = \sqrt{16\sin^2(t) + \cos^2(t) + 1}$

$$\vec{T}(t) = \frac{1}{||\vec{r}\,'(t)||}\vec{r}\,'(t) = \frac{1}{\sqrt{16\sin^2(t) + \cos^2(t) + 1}} < -4\sin(t),\ \cos(t),\ 1 >$$

$$\vec{T}(0) = \frac{1}{\sqrt{1+1}} < 0, 1, 1 >=< 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

$$T\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{4 + \frac{3}{4} + 1}} < -2,\ \frac{\sqrt{3}}{2},\ 1 >=< \frac{-4}{\sqrt{23}},\ \frac{\sqrt{3}}{\sqrt{23}},\ \frac{2}{\sqrt{23}} >$$

$$T\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{16+1}} < -4, 0, 1 >=< -\frac{4}{\sqrt{17}},\ 0,\ \frac{1}{\sqrt{17}} >$$



$$s = \int_0^{2\pi} \sqrt{(-4\sin(t))^2 + (\cos(t))^2 + 1}\ dt = 18.45342$$

the perimeter of the ellipse $\frac{x^2}{4} + y^2 = 1$: $\int_0^{2\pi} \sqrt{(-4\sin(t))^2 + (\cos(t))^2 + 0}\ dt = 17.15684$

2. Curvature

Let $s(t)$ be the length of arc of the curve C traced out by $\vec{r}(t) = < f(t),\ g(t),\ h(t) >$ for t be between a and b. Then

$$s(t) = \int_a^t \left|\left|\vec{r}\,'(u)\right|\right| du$$

The function $s(t)$ increases as t increases.

Example *Find the length of arc of* $\vec{r}(t) < \cos t,\ \sin t,\ t >$ *for* $t \geq 0$.

$$\vec{r}\,'(t) = < -\sin t,\ \cos t,\ 1 >,\quad \left|\left|\vec{r}\,'(t)\right|\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$s(t) = \int_0^t \left|\left|\vec{r}\,'(u)\right|\right| du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,u|_0^t = \sqrt{2}\,t$$

$$s(2\pi) = \sqrt{2}\,(2\pi) = 2\sqrt{2}\,\pi \quad (\textit{the circumference of a unit circle is } 2\pi).$$

The **curvature** κ of a curve C traced out by $\vec{r}(t)$ is

$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right|$$

Since

$$\vec{T}\,'(t) = \frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\left(\frac{ds}{dt}\right) \Rightarrow \frac{d\vec{T}}{ds} = \frac{\vec{T}\,'(t)}{\frac{ds}{dt}}$$

$$\frac{ds}{dt} = \frac{d}{dt}s(t) = \frac{d}{dt}\int_a^t \left|\left|\vec{r}\,'(u)\right|\right| du \overset{\text{Fundamental Theorem of Calculus}}{=} \left|\left|\vec{r}\,'(t)\right|\right|$$

So,

$$\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = \frac{1}{\left|\left|\vec{r}\,'(t)\right|\right|}\left|\left|\vec{T}\,'(t)\right|\right|$$

Example *Curvature of a line, a circle, and a helix.*

a. a line: $\vec{r}(t) = < x_0 + d_1 t,\ y_0 + d_2 t,\ z_0 + d_3 t >$

$$\vec{r}\,'(t) = < d_1,\ d_2,\ d_3 >,\quad \vec{T}(t) = \frac{1}{\sqrt{d_1^2 + d_2^2 + d_3^2}} < d_1,\ d_2,\ d_3 >,\quad \vec{T}\,'(t) = \vec{0}.$$

So, $\kappa = 0$, the curvature is zero, the curve is flat (no surprise).

b. a circle: $\vec{r}(t) = < b\cos t,\ b\sin t >$

$$\vec{r}\,'(t) = < -b\sin t,\ b\cos t >,\quad \vec{T}(t) = \frac{1}{b} < -b\sin t,\ b\cos t > = < -\sin t,\ \cos t >$$

$$\vec{T}\,'(t) = < -\cos t,\ -\sin t > = -\vec{r}(t)$$

$$\kappa = \frac{\left|\left|\vec{T}\,'(t)\right|\right|}{\left|\left|\vec{r}\,'(t)\right|\right|} = \frac{1}{b},$$ the curvature is a constant and is the reciprocal of the radius.

c. a helix: $\vec{r}(t) = < b\cos t,\ b\sin t,\ at >,\ a > 0,$ and $b > 0$

$$\vec{r}\,'(t) = < -b\sin t,\ b\cos t,\ a >,\ \vec{T}(t) = \frac{1}{\sqrt{b^2 + a^2}} < -b\sin t,\ b\cos t,\ 1 >$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{b^2 + a^2}} < -b\cos t,\ -b\sin t,\ 0 >$$

$$\kappa = \frac{\left\|\vec{T}\,'(t)\right\|}{\left\|\vec{r}\,'(t)\right\|} = \frac{\frac{b}{\sqrt{b^2+a^2}}}{\sqrt{b^2+a^2}} = \frac{b}{b^2+a^2} \text{ is also a constant.}$$

Another formula for curvature:

$$\kappa = \frac{\left\|\vec{r}\,'(t) \times \vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|^3}$$

a. an ellipse: $\vec{r}(t) = < a\cos t,\ b\sin t, 0 >,\ a > 0,\ b > 0$

$$\vec{r}\,'(t) = < -a\sin t,\ b\cos t,\ 0 >, \quad \vec{r}\,''(t) = < -a\cos t,\ -b\sin t,\ 0 >$$

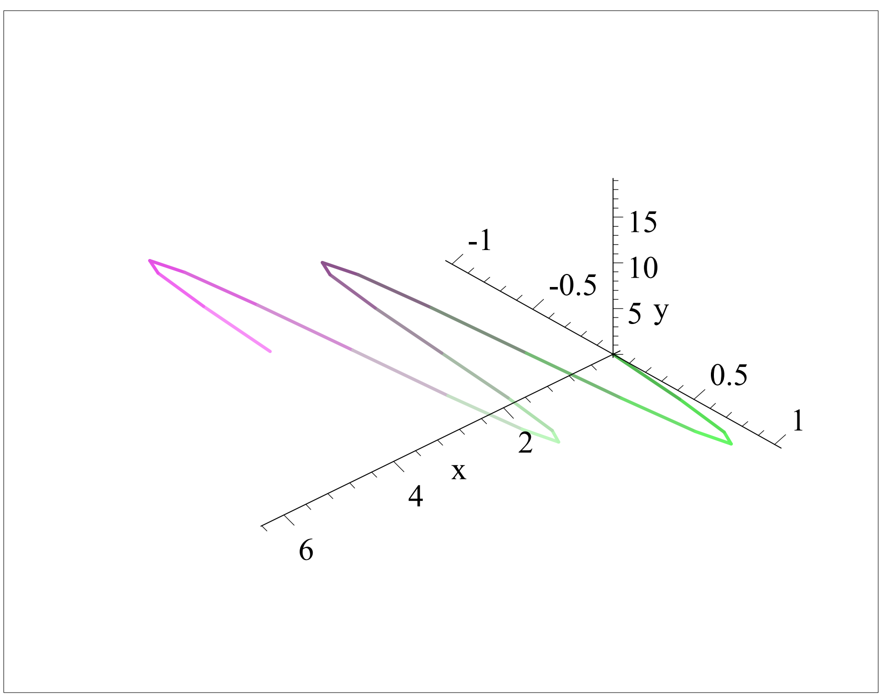
$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = (0)\vec{i} - (0)\vec{j} + \left(ab\sin^2 t + ab\cos^2 t\right)\vec{k} = ab\vec{k}$$

$$\kappa = \frac{ab}{\sqrt{\left(a^2\sin^2 t + b^2\cos^2 t\right)^3}}$$

$$\text{when } t = 0 \text{ and } t = \pi,\ \kappa = \frac{ab}{b^3} = \frac{a}{b^2}$$

$$\text{when } t = \frac{\pi}{2} \text{ and } t = \frac{3\pi}{2},\quad \kappa = \frac{ab}{a^3} = \frac{b}{a^2}$$

b. a sine curve in a plane: $\vec{r}(t) = < t, \sin(2t), 3t >,\ 0 \le t \le 2\pi$



$$\vec{r}\,'(t) = < 1, 2\cos(2t), 3 >,\ \vec{r}\,''(t) = < 0,\ -4\sin(2t),\ 0 >$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2\cos(2t) & 3 \\ 0 & -4\sin(2t) & 0 \end{vmatrix} = 12\sin(2t)\vec{i} - 4\sin(2t)\vec{k}$$

$$\kappa = \frac{\left\|\vec{r}\,'(t) \times \vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|^3} = \frac{\sqrt{144\sin^2(2t) + 16\sin^2(2t)}}{\left(\sqrt{10 + 4\cos^2(2t)}\right)^3} = \frac{\sqrt{150}\,|\sin(2t)|}{\left(\sqrt{10 + 4\cos^2(2t)}\right)^3}$$

$$\text{when } t = 0, \text{ and } t = \frac{\pi}{2},\ \kappa = 0, \quad \sin(2x) \approx 2x$$

$$\text{when } t = \frac{\pi}{4}, \text{ and } t = \frac{3\pi}{4},\ \kappa = \frac{\sqrt{150}}{\left(\sqrt{10}\right)^3} = \frac{\sqrt{15}}{10} \text{ - curvature is maximized}$$

c. a curve in xy-plane: $y = f(x) \Rightarrow \vec{r}(t) =< t,\ f(t),\ 0 >$

$$\vec{r}'(t) =< 1,\ f\,'(t),\ 0 >,\ \vec{r}\,''(t) =< 0,\ f\,''(t),\ 0 >$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f\,'(t) & 0 \\ 0 & f\,''(t) & 0 \end{vmatrix} = f\,''(t)\vec{k}$$

$$\kappa = \frac{\left\|\vec{r}\,'(t) \times \vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|^3} = \frac{\left|f\,''(t)\right|}{\left(\sqrt{1 + \left[f\,'(t)\right]^2}\right)^3} = \frac{\left|f\,''(x)\right|}{\left(\sqrt{1 + \left[f\,'(x)\right]^2}\right)^3}$$

Example *Find the curvature of the graph of $f(x) = \ln x$ at $x = 1$, $x \to \infty$ and $x \to 0^+$*

$$f\,'(x) = \frac{1}{x},\ f\,''(x) = -\frac{1}{x^2},\quad \kappa = \frac{\left|-\frac{1}{x^2}\right|}{\left(1 + \frac{1}{x^2}\right)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}}$$

$$\text{when } x = 1,\ \kappa = \frac{1}{2\sqrt{2}};\quad \lim_{x\to+\infty} \kappa = \lim_{x\to+\infty} \frac{|x|}{(x^2+1)^{3/2}} = 0;\quad \lim_{x\to0^+} \kappa = \lim_{x\to0^+} \frac{|x|}{(x^2+1)^{3/2}} = 0$$

d. curvature in polar coordinates: given $r = f(\theta)$, know that

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}.\ \text{Let } \vec{v}(\theta) =< \cos\theta,\ \sin\theta,\ 0 > \text{ and } \vec{r}(\theta) = f(\theta)\vec{v}(\theta).$$

$$\vec{v}\,'(\theta) < -\sin\theta,\ \cos\theta,\ 0 >,\quad \vec{v}\,''(\theta) =< -\cos\theta,\ -\sin\theta,\ 0 >= -\vec{v}(\theta)$$

$$\vec{r}\,'(\theta) = f\,'(\theta)\,\vec{v}(\theta) + f(\theta)\,\vec{v}\,'(\theta)$$

$$\vec{r}\,''(\theta) = f\,''(\theta)\,\vec{v}(\theta) + 2f\,'(\theta)\,\vec{v}\,'(\theta) + f(\theta)\,\vec{v}\,''(\theta)$$

$$\begin{aligned} \vec{r}\,'(\theta) \times \vec{r}\,''(\theta) &= \left(f\,'(\theta)\,\vec{v}(\theta) + f(\theta)\,\vec{v}\,'(\theta)\right) \times \left(f\,''(\theta)\,\vec{v}(\theta) + 2f\,'(\theta)\,\vec{v}\,'(\theta) + f(\theta)\,\vec{v}\,''(\theta)\right) \\ &= f\,'(\theta)f\,''(\theta)\,\vec{v}(\theta) \times \vec{v}(\theta) + f(\theta)f\,''(\theta)\ \vec{v}\,'(\theta) \times \vec{v}\,(\theta) + 2\left(f\,'(\theta)\right)^2 \vec{v}(\theta) \times \vec{v}\,'(\theta) \\ &\quad + 2f(\theta)f\,'(\theta)\,\vec{v}\,'(\theta) \times \vec{v}\,'(\theta) + f\,'(\theta)f(\theta)\,\vec{v}(\theta) \times \vec{v}\,''(\theta) + (f(\theta))^2\,\vec{v}\,'(\theta) \times \vec{v}\,''(\theta) \end{aligned}$$

$$\vec{v}(\theta) \times \vec{v}(\theta) = \vec{0}, \quad \vec{v}\,'(\theta) \times \vec{v}\,'(\theta) = 0, \quad \vec{v}\,(\theta) \times \vec{v}\,''(\theta) = 0 \text{ since } \vec{v}\,''(\theta) = -\vec{v}(\theta)$$

$$\vec{v}\,'(\theta) \times \vec{v}\,(\theta) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = -\vec{k}, \qquad \vec{v}(\theta) \times \vec{v}\,'(\theta) = \vec{k}$$

$$\vec{v}\,'(\theta) \times \vec{v}\,'' = -\vec{v}'(\theta) \times \vec{v}(\theta) = \vec{k},$$

$$\vec{r}\,'(\theta) \times \vec{r}\,''(\theta) = -f(\theta)f\,''(\theta)\vec{k} + 2\left(f\,'(\theta)\right)^2\vec{k} + (f(\theta))^2\vec{k} = \left(-f(\theta)f\,''(\theta) + 2\left(f\,'(\theta)\right)^2 + (f(\theta))^2\right)\vec{k}$$

$$\left|\left|\vec{r}\,'(\theta)\right|\right|^2 = \vec{r}\,'(\theta) \cdot \vec{r}\,'(\theta) = \left(f\,'(\theta)\,\vec{v}(\theta) + f(\theta)\,\vec{v}\,'(\theta)\right) \cdot \left(f\,'(\theta)\,\vec{v}(\theta) + f(\theta)\,\vec{v}\,'(\theta)\right)$$

$$= \left(f\,'(\theta)\right)^2\vec{v}(\theta) \cdot \vec{v}(\theta) + 2f(\theta)f\,'(\theta)\,\vec{v}(\theta) \cdot \vec{v}\,'(\theta) + \left(f(\theta)\right)^2\vec{v}\,'(\theta) \cdot \vec{v}\,'(\theta)$$

$$\vec{v}(\theta) \cdot \vec{v}(\theta) = 1, \quad \vec{v}(\theta) \cdot \vec{v}\,'(\theta) = 0, \quad \vec{v}\,'(\theta) \cdot \vec{v}\,'(\theta) = 1$$

$$\left|\left|\vec{r}\,'(\theta)\right|\right|^2 = \left(f\,'(\theta)\right)^2 + \left(f(\theta)\right)^2$$

$$\kappa = \frac{\left|-f(\theta)f\,''(\theta) + 2\left(f\,'(\theta)\right)^2 + (f(\theta))^2\right|}{\left(\left(f\,'(\theta)\right)^2 + \left(f(\theta)\right)^2\right)^{3/2}}$$

Example *Find the curvature of the polar curve* $r = 1 - 2\sin\theta$, *at* $\theta = 0,\ \theta = \pi$

$$f(\theta) = 1 - 2\sin\theta,\ f\,'(\theta) = -2\cos\theta,\ f\,''(\theta) = 2\sin\theta$$

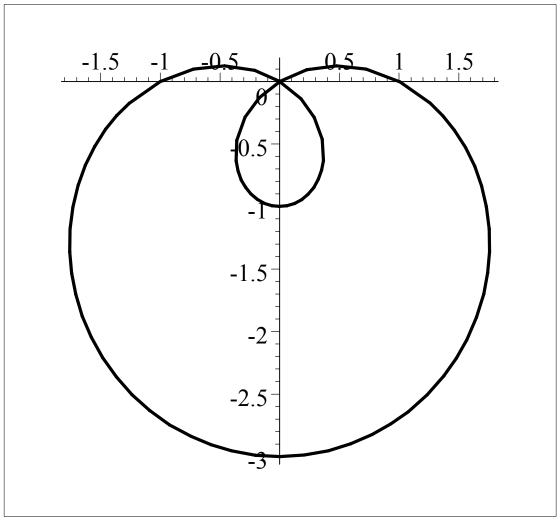
$$\kappa = \frac{\left|-(1-2\sin\theta)(2\sin\theta) + 2(4\cos^2\theta) + (1-2\sin\theta)^2\right|}{\left(4\cos^2\theta + (1-2\sin\theta)^2\right)^{3/2}}$$

$$= \frac{|-2\sin\theta + 4\sin^2\theta + 8\cos^2\theta + 1 - 4\sin\theta + 4\sin^2\theta|}{\left(4\cos^2\theta + 1 - 4\sin\theta + 4\sin^2\theta\right)^{3/2}} = \frac{|9 - 6\sin\theta|}{(5 - 4\sin\theta)^{3/2}}$$

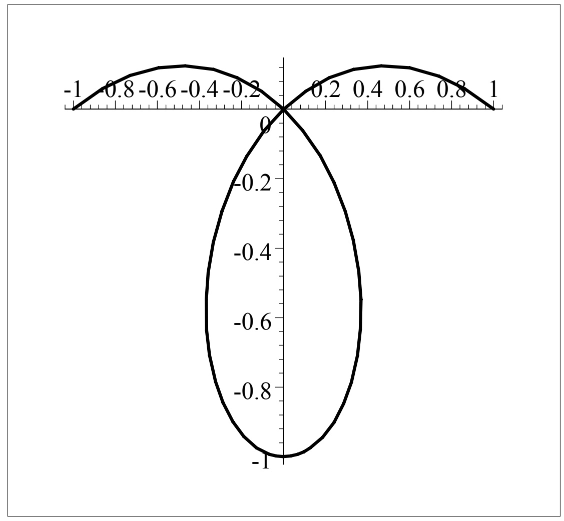
$$\text{when } \theta = 0,\ \theta = \pi,\ \kappa = \frac{9}{(5)^{3/2}} = 0.8049845$$

$$\text{when } \theta = \frac{\pi}{2},\quad \kappa = 3$$

$$\text{when } \theta = \frac{3\pi}{2},\quad \kappa = \frac{15}{9} = 1.666667$$



$r = 1 - 2\sin\theta, \; -\pi \le \theta \le \pi$



$r = 1 - 2\sin\theta, \; 0 \le \theta \le \pi$