

Absolute Convergence and Ratio Test - 8.5

1. Absolute Convergence:

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Note that if $a_n \geq 0$ for all n , and the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the series is absolutely convergent.

Example Consider if the alternating series is absolute convergent.

$$a. \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \quad b. \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p} \quad c. \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \quad d. \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{3}{\pi}\right)^n$$

a. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Since $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is not absolutely convergent (in some textbooks, it is called convergent conditionally).

b. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$

Know that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is convergent for any $p > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent only if $p > 1$. So, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is absolutely convergent if $p > 1$.

c. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

$|a_n| = \left| \frac{\cos(n)}{n^2} \right| \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right|$ is convergent by the comparison test. So, the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent.

d. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{3}{\pi}\right)^n = \sum_{n=1}^{\infty} \left(\frac{3}{\pi}\right) \left(-\frac{3}{\pi}\right)^{n-1}$

It is a convergent geometric series with $r = -\frac{3}{\pi}$ and $a = \frac{3}{\pi}$.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{3}{\pi}\right)^n = \frac{3}{\pi} \left(\frac{1}{1 + \frac{3}{\pi}} \right) = \frac{3}{\pi + 3}$$

It is also convergent absolutely.

2. Conditionally Convergent Series:

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Example p -series for $p \leq 1$ are convergent conditionally.

3. The Ratio Test:

Let $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

a. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

b. If $L > 1$ (or $L = \infty$), then the series $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $L = 1$, then there no conclusion from this test.

Example Determine if the series is absolutely convergent.

$$a. \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2^{n+1}} \quad b. \sum_{n=1}^{\infty} \frac{n^n}{n!} \quad c. \sum_{n=1}^{\infty} \frac{(n+2)!}{n!4^n} \quad d. \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

a. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{2^{n+2}}}{(-1)^n \frac{n^3}{2^{n+1}}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^3}{n^3} \frac{1}{2} = \frac{1}{2} < 1$$

So, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2^{n+1}}$ is absolutely convergent by the ratio test.

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e > 1$$

So, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the ratio test.

c. $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!4^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+3)!}{(n+1)!4^{n+1}} \frac{n!4^n}{(n+2)!} = \lim_{n \rightarrow \infty} \frac{n+3}{4(n+1)} = \frac{1}{4} < 1$$

So, the series $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!4^n}$ is absolutely convergent by the ratio test.

d. $\sum_{n=1}^{\infty} \frac{n!3^n}{(2n)!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!3^{k+1}}{(2n+2)!} \frac{(2n)!}{n!3^k} = \lim_{n \rightarrow \infty} \frac{3(n+1)}{(2n+2)(2n+1)} = 0 < 1$$

So, the series $\sum_{n=1}^{\infty} \frac{n!3^n}{(2n)!}$ is absolutely convergent by the ratio test.

Example A series $\sum_{n=1}^{\infty} a_n$ is defined by the equations:

$$a_1 = 1, \quad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$$

Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Check:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 + \cos n}{\sqrt{n}} \right| = 0 < 1$$

By the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges.

4. The Root Test:

Let $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$.

- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $L > 1$ (or $L = \infty$), then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$, then there no conclusion from this test.

Example Use the Root Test to determine the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n-4} \right)^n$.

Check

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{2n+3}{5n-4} \right|^n} = \lim_{n \rightarrow \infty} \frac{2n+3}{5n-4} = \frac{2}{5} < 1.$$

So the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n-4} \right)^n$ is a convergent series.