The Integral and Comparison Test for Series and Estimate Sums - 8.3

1. The Integral Test:

Suppose that f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. That is

a. If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

b. If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example *Determine the convergence of the series using the integral test.*

$$a.\ \sum_{n=1}^{\infty}\frac{1}{n}\quad b.\ \sum_{n=1}^{\infty}\frac{1}{n^2}\quad c.\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\quad d.\ \sum_{n=1}^{\infty}\frac{1}{n^p},\ p>0$$

$$e.\ \sum_{n=1}^{\infty}\frac{\ln n}{n}\quad f.\ \sum_{n=2}^{\infty}\frac{1}{n(\ln n)}\quad g.\ \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

a. $\sum_{n=1}^{\infty}\frac{1}{n}$, let $f(x) = \frac{1}{x}$ for $x \geq 1$. Check the convergence of the improper integral

$$\int_1^{\infty}\frac{1}{x}dx = \lim_{t\to\infty}\ln|x|\Big|_1^t = \lim_{t\to\infty}\ln|t| - 0 = \infty$$

Since $\int_1^{\infty}\frac{1}{x}dx$ diverges, $\sum_{n=1}^{\infty}\frac{1}{n}$ also converges.

b. $\sum_{n=1}^{\infty}\frac{1}{n^2}$, let $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Check the convergence of the improper integral

$$\int_1^{\infty}\frac{1}{x^2}dx = \lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t = -\lim_{t\to\infty}\left(\frac{1}{t} - 1\right) = 1$$

Since $\int_1^{\infty}\frac{1}{x^2}dx$ converges, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ also converges. But we don't know the limit.

c. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, let $f(x) = \frac{1}{\sqrt{x}}$ for $x \geq 1$. Check the convergence of the improper integral

$$\int_1^{\infty}\frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^t x^{-1/2}dx = \lim_{t\to\infty}(2)x^{1/2}\Big|_1^t = 2\left[\lim_{t\to\infty}\sqrt{t} - 1\right] = \infty$$

Since $\int_1^{\infty}\frac{1}{\sqrt{x}}dx$ diverges, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ also converges.

d. $\sum_{n=1}^{\infty}\frac{1}{n^p}$, $p > 0$. Let $f(x) = \frac{1}{x^p}$ for $x \geq 1$. Check the convergence of the improper integral

$$\int_1^{\infty}\frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty}\begin{cases}\ln|x|\Big|_1^t & \text{if } p = 1\\ \frac{1}{1-p}x^{1-p}\Big|_1^t & \text{if } 0 < p < 1\\ \frac{-1}{p-1}\left(\frac{1}{x^{p-1}}\right)\Big|_1^t & \text{if } 1 < p\end{cases}$$

$$= \lim_{t\to\infty}\begin{cases}\ln|t| - 0 & \text{if } p = 1\\ \frac{1}{1-p}[t^{1-p} - 1] & \text{if } 0 < p < 1\\ \frac{-1}{p-1}\left(\frac{1}{t^{p-1}} - 1\right)\Big|_1^t & \text{if } 1 < p\end{cases}$$

$$= \begin{cases}\text{diverges} & \text{if } p = 1\\ \text{diverges} & \text{if } 0 < p < 1\\ \frac{1}{p-1},\ \text{converges} & \text{if } 1 < p\end{cases}$$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$ is called **a** p **–series**. So, **a** p **–series converges if** $p > 1$.

e. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$, let $f(x) = \frac{\ln x}{x}$ for $x \geq 1$. Check the convergence of the improper

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x} dx \underset{x = 1, u = 0, x = t, u = \ln t}{\overset{u = \ln x, du = \frac{1}{x}dx}{=}} \lim_{t\to\infty} \int_0^{\ln t} u du$$

$$= \lim_{t\to\infty} \frac{1}{2} u^2 |_0^{\ln t} = \frac{1}{2}\left[\lim_{t\to\infty} (\ln t)^2 - 0\right] = \infty$$

Since $\int_1^{\infty} \frac{\ln x}{x} dx$ diverges, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

f. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$, let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. Check the convergence of the improper

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x} dx \underset{x = 2, u = \ln 2, x = t, u = \ln t}{\overset{u = \ln x, du = \frac{1}{x}dx}{=}} \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u} du$$

$$= \lim_{t\to\infty} \ln|u| |_{\ln 2}^{\ln t} = \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty$$

Since $\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ also diverges.

g. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, let $f(x) = \frac{1}{x(\ln x)^2}$ for $x \geq 2$. Check the convergence of the improper

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^2} dx \underset{x = 2, u = \ln 2, x = t, u = \ln t}{\overset{u = \ln x, du = \frac{1}{x}dx}{=}} \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du$$

$$= \lim_{t\to\infty} \left(-\frac{1}{u}\right) |_{\ln 2}^{\ln t} = -\lim_{t\to\infty} \frac{1}{\ln t} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

Since $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ converges, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

2. **Comparison Test:**
   - **a. Comparison test:**

     Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **series with positive terms.** Then

     i. If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for $n \geq n_0 > 0$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

     ii. If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all $n \geq n_0 > 0$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.
   - **b. The limit comparison test:**

     Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **series with positive terms.** If

     $$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } c \text{ is a finite number and } c > 0,$$

     then either both series converge and both diverge.

Example *Determine the convergence of series using a comparison test.*

$$a.\ \sum_{n=1}^{\infty} \frac{2n^2 + n + 100}{n^3 - 2} \quad b.\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + n + 1}} \quad c.\ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + n} \quad d.\ \sum_{n=1}^{\infty} \frac{1}{(\sqrt{n} + 1)^2}$$

$$e.\ \sum_{n=1}^{\infty} \frac{tan^{-1}(2n)}{n^2} \quad f.\ \sum_{n=1}^{\infty} \frac{2 + sin(n)}{n}$$

a. $\sum_{n=1}^{\infty} \frac{2n^2+n+100}{n^3-2}$, compare $a_n = \frac{2n^2+n+100}{n^3-2}$ with $b_n = \frac{2n^2}{n^3} = \frac{2}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+n+100}{n^3-2}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{2n^3+n^2+100n}{2(n^3-2)} = \frac{2}{2} = 1 = c > 0$$

and $\sum_{n=1}^{\infty} \frac{2}{n}$, a harmonic series, diverges, $\sum_{n=1}^{\infty} \frac{2n^2+n+100}{n^3-2}$ also diverges.

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n+1}}$, compare $a_n = \frac{1}{\sqrt{n^2+n+1}}$ with $\frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+n+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} = 1 = c > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$, a harmonic series, diverges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n+1}}$ also diverges.

c. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$, compare $a_n = \frac{\sqrt{n}}{n^2+n}$ with $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+n}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1 = c > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p –series with $p > 1$, is convergent , $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$ is also convergent.

d. $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n}+1)^2}$, compare $a_n = \frac{1}{(\sqrt{n}+1)^2}$ with $b_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{\frac{1}{(\sqrt{n}+1)^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(\sqrt{n}+1)^2} = \lim_{n\to\infty} \frac{n(\frac{1}{n})}{(\sqrt{n}+1)^2(\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^2} = 1 = c > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$, a harmonic series, diverges, $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n}+1)^2}$ is divergent.

e. $\sum_{n=1}^{\infty} \frac{\tan^{-1}(2n)}{n^2}$, since $n \geq 1$

$$0 \leq \tan^{-1}(2n) < \frac{\pi}{2} \Rightarrow \frac{\tan^{-1}(2n)}{n^2} < \frac{\pi}{2}\left(\frac{1}{n^2}\right).$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p –series with $p = 2$ converges, $\sum_{n=1}^{\infty} \frac{\tan^{-1}(2n)}{n^2}$ also converges.

f. $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$, since

$$1 \leq 2+\sin(n) \leq 3 \Rightarrow \frac{2+\sin(n)}{n} \geq \frac{1}{n}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$, a harmonic series, diverges, $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$ diverges.

3. Estimating the Sum of Series:

Assume $a_n \geq 0$ for $n \geq 1$ and the series $\sum_{n=1}^{\infty} a_n$ converges with the sum S. Let $S_m = \sum_{n=1}^{m} a_n$ be the mth partial sum of $\sum_{n=1}^{\infty} a_n$. Then $\lim_{m\to\infty} S_m = S$. Define

$$R_m = S - S_m = a_{m+1} + a_{m+2} + \ldots.$$

R_m is called the mth remainder and is the true error made when $\sum_{n=1}^{\infty} a_n \approx S_m$. How to estimate R_m? Let $f(n) = a_n$. Observe that:

$$\int_{m+1}^{\infty} f(x)dx \leq a_{m+1} + a_{m+2} + \ldots \leq \int_{m}^{\infty} f(x)dx \Rightarrow \int_{m+1}^{\infty} f(x)dx \leq R_m \leq \int_{m}^{\infty} f(x)dx.$$

$$R_m = S - S_m \Rightarrow \int_{m+1}^{\infty} f(x)dx \leq S - S_m \leq \int_{m}^{\infty} f(x)dx \Rightarrow S_m + \int_{m+1}^{\infty} f(x)dx \leq S \leq S_m + \int_{m}^{\infty} f(x)dx.$$

From the above inequalities of S and R_m, we may improve the approximation of S by

$$S \approx S_m + \frac{1}{2}\left(\int_{m+1}^{\infty} f(x)dx + \int_{m}^{\infty} f(x)dx\right)$$

and better estimate R_m by

$$R_m \approx \frac{1}{2}\left(\int_{m+1}^{\infty} f(x)dx + \int_{m}^{\infty} f(x)dx\right)$$

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Example *Consider the series* $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

- **a.** *Show that the series converges.*
- **b.** *Approximate the sum of the series by using* S_{10} *and estimate the approximation error.*
- **c.** *How many terms are required to ensure that the sum is accurate to within* 0.0005*?*

a. It is a p –series with $p = 3 > 1$, so it converges.

b.

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \sum_{n=1}^{10} \frac{1}{n^3} = 1.197532$$

$$0.0041322 = \int_{11}^{\infty} \frac{1}{x^3}dx \leq R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}dx = 0.005 \Rightarrow R_{10} < 0.005$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.1975 + \frac{1}{2}(0.005 + 0.0041322) = 1.202066$$

For checking to see how good our improvement is:

$$\sum_{n=1}^{100} \frac{1}{n^3} = 1.202007$$

c.

$$R_n \leq \int_{n}^{\infty} \frac{1}{x^3}dx = \frac{1}{2n^2}$$

Find n such that $R_n < 0.0005$.

$$\frac{1}{2n^2} < 0.0005 \Rightarrow n^2 > \frac{1}{(2)(0.0005)} = 1000.0 \Rightarrow n > \sqrt{1000} = 31.623$$

Let $n = 32$ and $S_{32} = \sum_{n=1}^{32} \frac{1}{n^3} = 1.201584$.

Example *Use the* S_{100} *to approximate the sum of the series* $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ *and estimate the approximation error.*

$S_{100} = \sum_{n=1}^{100} \frac{1}{n^3+1} = 0.68645$. Error R_{100} can be estimated by

$$R_{100} \leq \int_{101}^{\infty} \frac{1}{x^3+1}dx \leq \int_{101}^{\infty} \frac{1}{x^3}dx = \frac{1}{2(101)^2} = 0.000049015$$