Exponential Growth and Decay Problems - (6.4)-(6.5)

1. **Differential Equations:**
   A differential equation is an equation that contains an unknown function and some of its derivatives.

2. **Growth and Decay Problems:**
   Let $y(t)$ represent the number of bacteria in culture at the time t. Then the rate of change of bacterial population is $y'(t)$. If we know that $y'(t)$ is proportional to $y(t)$, i.e.

$$y'(t) = ky(t), \text{ and } y(0) = y_0.$$

then y can be solved as follows:

$$\frac{dy}{y} = kdt \Leftrightarrow \int \frac{dy}{y} = \int kdt \Leftrightarrow \ln|y| = kt + C \Leftrightarrow e^{\ln|y|} = e^{kt+C}$$

$$y = Ce^{kt},\ y(0) = Ce^0 = y_0,\quad y = y_0e^{kt}$$

k is called a growth or decay constant.

Note that the solution of the differential equation of the form: $y' = ky$ is $y = Ce^{kt}$. If $k > 0$, y grows exponentially; if $k < 0$, y decreases exponentially.

Example *Let P be the population of a city. Assume that the growth rate of the population of this city is proportional to the size of the population. If we know the population of the city was 35,000 in 1960 and was 37,000 in 1990. What is the population of the city in 2002?*

a. Set up a relation between the population and the rate of change of the population:
Let $P(t)$ be the population of this city at the time t, where $t = 0$ represents 1960. Then $P'(t)$ is the rate of change of the population at the time t. Since the growth rate of the population of this city is proportional to the size of the population,

$$P'(t) = kP(t).$$

This is a differential equation.

b. Solve $P(t)$: From the derivation above, we know

$$P(t) = P_0e^{kt}$$

c. Find the constants P_0 and k :
We know $P(0) = 35000$ and $P(30) = 37000$. Use these two conditions to find k and C.

$$P(0) = P_0 = 35000$$

$$P(30) = 35000e^{30k} = 37000 \Rightarrow e^{30k} = \frac{37000}{35000} = \frac{37}{35} \Rightarrow \ln e^{30k} = \ln\frac{37}{35} \Rightarrow 30k = \ln\frac{37}{35},$$

$$k = \frac{1}{30}\ln\frac{37}{35} = 0.0018523$$

$$P(t) = 35000e^{0.0018523t}.$$

d. The population in 2002 is $P(42) = 35000e^{0.0018523(42)} = 37832$

Example *The half-life of radium-226 is 1590 years.*

a. A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of radium-226 that remains after t years.

b. Find the mass after 1000 years correct to the nearest milligram.

c. When will the mass be reduced 30mg?

a. Let P be the mass of radium-226. Then

$$P(t) = P_0e^{kt}.$$

The half-live is 1590 implies that

$$\frac{1}{2}C = P_0e^{k(1590)} \Rightarrow \frac{1}{2} = e^{1590k} \Rightarrow -\ln 2 = 1590k,\ k = \frac{-\ln 2}{1590} = -0.00043594$$

$$P(t) = 100e^{-0.0004359t}$$

b. $P(1000) = 100e^{-0.0004359(1000)} = 64.668$ mg

c. Find t such that $P(t) = 30$.

$$100e^{-0.0004359t} = 30,\ e^{-0.0004359t} = \frac{30}{100} = 0.3,\ -0.00043594t = \ln 0.3$$

$$t = \frac{\ln 0.3}{-0.00043594} = 2761.8 \text{ years}$$

3. Newton's Law of Cooling:

When a hot object is introduced into cool surroundings, the rate at which the object cools is not proportional to its temperature, but rather, is proportional to the difference in temperature between the object and the surroundings. Let $T(t)$ be the temperature of the object at the time t, T_0 be the initial temperature of the object, and T_s be the temperature of the surroundings. Then

$$T'(t) = k(T(t) - T_s),\ T(0) = T_0$$

Solve for $T(t)$:

$$\frac{dT}{dt} = k(T - T_s),\quad \frac{1}{T - T_s}dT = kdt \Rightarrow \int \frac{1}{T - T_s}dT = \int kdt$$

$$\ln|T - T_s| = kt + C,\ T - T_s = e^{kt+C} = Ce^{kt},\quad T(t) = T_s + Ce^{kt}$$

$$T(0) = T_s + C = T_0,\ C = T_0 - T_s$$

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

Example *A small bowl of rice soup served at $200°F$ cools to $160°F$ in 1 minute in a $70°F$ room. What temperature will this rice soup be when the bowl has reached $120°F$?*

We know: $T_s = 70°F,\ T_0 = 200°F,\ T(1) = 160°F,$ and want to find out when $T(t) = 120°F$.

$$T(t) = 70 + (200 - 70)e^{kt} = 70 + 130e^{kt},\ T(1) = 70 + 130e^{k} = 160,$$

$$e^k = \frac{160 - 130}{70} = \frac{3}{7},\ k = \ln\left(\frac{3}{7}\right)$$

$$T(t) = 70 + 130e^{t\ \ln(3/7)},\ T(t) = 120 \Rightarrow 70 + 130e^{t\ \ln(3/7)} = 120$$

$$e^{t\ln(3/7)} = \frac{120 - 70}{130} = \frac{5}{13},\ t\ln\left(\frac{3}{7}\right) = \ln\left(\frac{5}{13}\right),\ t = \frac{\ln 5 - \ln 13}{\ln 3 - \ln 7} = 1.127716$$

Example *At 9:30am, a secret agent was found murdered in a $75°F$ room. The body temperature was $90°F$ and was $85°F$ at 10:00am. When was the time of death?*

We know that $T_0 = 90°F,\ T_s = 75°F,\ T(0.5) = 85°F$. Find the time when $T = 98.6°F$.

$$T(t) = 75 + 15e^{kt},\ T(0.5) = 75 + 15e^{0.5k} = 85,\quad e^{0.5k} = \frac{10}{15} = \frac{2}{3}$$

$$0.5k = \ln\left(\frac{2}{3}\right),\ k = \frac{1}{0.5}\ln\left(\frac{2}{3}\right) = 2\ln\left(\frac{2}{3}\right)$$

$$T(t) = 75 + 15e^{2\ln(2/3)\ t} = 98.6,\quad e^{2\ln(2/3)\ t} = \frac{98.6 - 75}{15} = \frac{23.6}{15}$$

$$t = \frac{\ln 23.6 - \ln 15}{2(\ln 2 - \ln 3)} = -0.5588601 \text{ hour } = -0.5588601(60) = -33.53161 \text{ minutes}$$

The time of death is about 8:57am.

4. Logistic Growth:

Let P be a population. Then $\frac{1}{P}\left(\frac{dP}{dt}\right)$ is the relative growth (or decay) rate. In an exponential grow or decay model,

$$\frac{1}{P}\frac{dP}{dt} = k \text{ a constant.}$$

A logistic differential equation:

$$\frac{1}{P}\frac{dP}{dt} = k - k\left(\frac{P}{K}\right) \Rightarrow \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right), \text{ where } K \text{ is a constant.}$$

Note that:

a. If $\frac{P}{K} \approx 0$, then $\frac{dP}{dt} \approx kP$.

b. If $\frac{P}{K} \approx 1$, then $\frac{dP}{dt} \approx 0$.

c. Assume $k > 0$.
If $\frac{P}{K} < 1$, then $\frac{dP}{dt} > 0$ and P is increasing; if $\frac{P}{K} > 1$, then $\frac{dP}{dt} < 0$ and P is decreasing.
K is called the carrying capacity of P

Solution of a logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{k}{K}P(K-P) \Rightarrow \frac{K}{P(K-P)}dP = kdt$$

$$\int \frac{K}{P(K-P)}dP = \int kdt = kt + C$$

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P} \quad \text{(a quick verification)}$$

$$\int \frac{K}{P(K-P)}dP = \int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \ln|P| - \ln|K-P| + C = \ln\left|\frac{P}{K-P}\right| + C$$

$$\ln\left|\frac{P}{K-P}\right| = kt + C \Rightarrow \frac{P}{K-P} = e^{kt+C} = Ce^{kt} \Rightarrow \frac{K-P}{P} = \frac{1}{Ce^{kt}} = Ce^{-kt}$$

$$\frac{K}{P} - 1 = Ce^{-kt} \Rightarrow \frac{K}{P} = Ce^{-kt} + 1 \Rightarrow P(t) = \frac{K}{1+Ce^{-kt}}$$

The solution of the equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ is $P(t) = \frac{K}{1+Ce^{-kt}}$.

Example *a. Find the solution of the initial value problem:*

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right), \; P(0) = 100.$$

b. Find P when t = 40 and t = 80.

c. At what time does the population reach 900?

a. $P(t) = \frac{1000}{1+Ce^{-0.08t}}$. Solve C using the condition $P(0) = 100$.

$$P(0) = \frac{1000}{1+C} = 100 \Rightarrow 1 + C = \frac{1000}{100} = 10 \Rightarrow C = 10 - 1 = 9$$

$$P(t) = \frac{1000}{1+9e^{-0.08t}}$$

b. $P(40) = \frac{1000}{1+9e^{-0.08(40)}} = 731.6$, $P(80) = \frac{1000}{1+9e^{-0.08(80)}} = 985.27$

c. Find t such that $P(t) = 900$.

$$\frac{1000}{1+9e^{-0.08t}} = 900 \Rightarrow 1+9e^{-0.08t} = \frac{1000}{900} = \frac{10}{9}$$

$$\frac{10}{9} - 1 = 9e^{-0.08t} \Rightarrow \frac{1}{9} = 9e^{-0.08t} \Rightarrow e^{-0.08t} = \frac{1}{81}$$

$$-0.08t = \ln\frac{1}{81} = -\ln 81 \Rightarrow t = \frac{\ln 81}{0.08} = 54.931$$