Stability of Two-Dimensional Discrete Systems With Periodic Coefficients

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Abstract—Two-dimensional (2-D) discrete systems with periodic coefficients are considered for stability. These systems are called periodically shift variant (PSV) digital filters and have many applications in signal processing that include the filtering of 2-D signals with cyclostationary noise, scrambling of digital images, and implementation of multirate filter banks. In this paper, the filters are formulated in the form of the well known Fornasini–Marchesini state-space model with periodic coefficients. This PSV model is then studied for stability. Two sufficient conditions and one necessary condition are established for asymptotic stability. Some examples are given to illustrate the results.

I. INTRODUCTION

IN MANY FIELDS of engineering, cyclostationary signals are often encountered. A cyclostationary process is one which has the property that its statistics vary periodically with time. All forms of bauded data transmission exhibit cyclostationary characteristics. Amplitude-modulated signals can also be modeled as cyclostationary if carrier synchronization is obtained. Also, the statistics of video signals exhibit a periodicity at the horizontal line rate. A detailed discussion of cyclostationarity in communications and signal processing can be found in [1]. Filtering of cyclostationary signals requires a filter which has a periodically varying impulse response. Periodically shift variant (PSV) filters have been shown to significantly improve estimation of cyclostationary signals in noise [2], [3]. PSV filters are also effective in filtering signals with cyclostationary additive noise. Other applications of PSV filters include speech scrambling [4] and decimator-interpolator filter design [5].

One-dimensional PSV filters have received widespread attention in the literature over the past two decades. Design of PSV filters has been discussed in [4]–[8]. Implementation aspects of these filters have been investigated in [9]–[11]. However, very little attention has been given to two-dimensional (2-D) PSV systems. These filters have applications in processing digital video images with cyclostationary noise and in the design of image scramblers. They are also useful in the analysis, design, and implementation of 2-D multirate filter banks. In [12], 2-D PSV filters have been analyzed in direct form. 2-D state-space PSV filters have been considered in [13], where equivalent shift-invariant block structures were derived. These structures are useful for analysis, but are computationally intensive for implementation. In [14] and [15], 2-D PSV systems were considered and new shift-invariant structures were derived. These structures are computationally efficient and lend themselves to easier analysis.

In this paper, 2-D state-space PSV filters are formulated and analyzed for stability. Sufficient conditions for stability are derived using energy functions in the spatial domain only. This is in contrast to the Z-domain approach used in [16]. The paper is organized as follows. In Section II, the system is described and formulated by the second model of Fornasini–Marchesini (FM). Based on the model, some definitions and observations are given in Section III. Two sufficient conditions and one necessary condition for stability are established in Section IV. Examples illustrating the results are given in Section V. Conclusions are made in Section VI.

II. SYSTEM DESCRIPTION

The difference equation representation for a linear PSV 2-D discrete system can be written in the form

\[
y(i,j) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} (i,j) y(i - m, j - n) + \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} b_{mn} (i,j) u(i - m, j - n) \tag{1}
\]

where \((m,n) \neq (0,0)\) for \(a_{mn}\). The coefficients are periodically shift variant, i.e.,

\[
a_{mn} (i,j) = a_{mn} (i + P; j) = a_{mn} (i,j + Q)
\]

\[
b_{mn} (i,j) = b_{mn} (i + P; j) = b_{mn} (i,j + Q)
\]

where the period is \((P; Q)\) and \(P; Q\) are positive integers, not both zero.

As in LSIV systems, the above 2-D PSV system can be represented in several different state-space forms with periodic coefficient matrices [16]. The second model of FM [17] is adopted here with \(a_{mn} (i,j)\) and \(b_{mn} (i,j)\) being the periodic coefficients. The second model of FM can be written as follows:

\[
x(i+1,j+1) = A(i+1,j+1)x(i+1,j) + B(i+1,j+1)x(i,j) + D(i+1,j+1)u(i,j)
\]

\[
y(i,j) = E(i,j)x(i,j) + F(i,j)u(i,j) \tag{2}
\]

\[
A(i+1,j+1) = A(i+1,j+1)
\]

\[
B(i+1,j+1) = B(i+1,j+1)
\]

\[
D(i+1,j+1) = D(i+1,j+1)
\]

\[
E(i,j) = E(i,j)
\]

\[
F(i,j) = F(i,j)
\]

Manuscript received October 24, 1996; revised October 6, 1997. The work of T. Bose, K. S. Joo, and G.-F. Xu was supported in part by a grant from the Colorado Advanced Software Institute. The work of M.-Q. Chen was supported in part by a grant from the Citadel Development Foundation. This paper was recommended by Associate Editor B. A. Shenoi.

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Publisher Item Identifier S 1057-7130(98)05053-8.
where \( x(i, j) \in \mathbb{R}^{L \times 1} \), \( A(i, j), B(i, j) \in \mathbb{R}^{L \times L} \), \( D(i, j) \in \mathbb{R}^{L \times 1} \), \( y(i, j), u(i, j), P(i, j) \in \mathbb{R}^{1 \times 1} \), and \( E(i, j) \in \mathbb{R}^{1 \times L} \).

The coefficient matrices are functions of \( a_{nm}(i, j) \) and \( b_{nm}(i, j) \) in (1) and periodically shift variant with period \((P, Q)\). In this paper, the zero-input stability of the state equation (2) is studied, and therefore, only the periodic coefficient matrices \( A(i, j) \) and \( B(i, j) \) are considered.

### III. Observations and Definitions

The observations on periodic matrices and definitions of the index sets for the coefficient matrices, the permutation matrices, and matrices \( C_{p,q}^{(i,j)} \) given in this section will be used in deriving the stability conditions in the next section.

Consider the first quadrant of state variables \( x(i, j) \) whose coefficient matrices are assumed to be periodic with period \((P, Q)\), i.e., \( A(i, j) = A(i+P, j) = A(i, j+Q) \) and \( B(i, j) = B(i+P, j) = B(i, j+Q) \). In Fig. 1, each corner of the grid is denoted by an index pair \((i, j)\), which represents the coefficient matrices \( A(i, j) \) and \( B(i, j) \) used for the computation of the state variable at that corner of the grid. Throughout the paper, let \( P = Sp \) and \( Q = Sq \) where \( p, q \) are positive integers, and \( S \) is the greatest common divisor of \( P \) and \( Q \). Also, let \( R = Spq \) be the least common denominator of \( P \) and \( Q \).

Notation \( \rho(\cdot) \) denotes the spectral radius of a matrix and \( (\cdot)^T \) denotes the transpose of a matrix or a vector. For \( X = \{x_{ij}\}, Y = \{y_{ij}\}, X \geq Y \), means \( x_{ij} \geq y_{ij}, \forall i, j \) and \( |X| = \{|x_{ij}|\} \).

Also, \( E = \max\{E_i, E_i \in \mathbb{R}^{L \times F}\} \) means \( E \geq E_i, \forall i \).

**Observation (1):** The state variables \( x(i, j) \) in every block of size \( P \times Q \) use the same set of coefficient matrices \( A(i, j) \) and \( B(i, j) \).

For example, in Fig. 1, the state variables in the blocks \((4 \times 6)\) denoted by #1 and #2 use the same set of coefficient matrices.

**Observation (2):** The state variables \( x(i, j) \) along each diagonal depend on a set of only \( R \) pairs of coefficient matrices \( A(i, j) \) and \( B(i, j) \) when they are related to the state variables on the previous diagonal.
For example, each of the first and second diagonal in Fig. 1 uses a set of $R = 12$ pairs of coefficient matrices.

**Observation (3):** In total, there are $S$ different sets of $R$ pairs of coefficient matrices $A(i,j)$ and $B(i,j)$.

For example, in Fig. 1 there are $S = 2$ different sets of 12 pairs of coefficient matrices, as shown on the two consecutive diagonals.

Observation (1) can be easily verified by Fig. 1. Detailed verifications for Observations (2) and (3) are given in the Appendix.

To derive the stability conditions, we need the following definitions.

**Definition (1):** Index Sets.

Define index sets for coefficient matrices used by the state variables along the diagonal as the first index decreases, as follows:

$$I_r = \{(r;0),(r-1,1),\ldots,(r-Q+1,Q-1),$$
$$\ldots,(r-(p-1)Q,0),(r-(p-1)Q-1,1),\ldots,(r-2Q+1,Q-1),$$
$$\ldots,(r-(p-1)Q+1,Q-1)\}$$

(without assigning each first index to an integer in the congruence class mod $P$),

$$I_r = \{(j,m)\text{ where } r-nQ-m \equiv j \text{ mod } P, 0 \leq j \leq P-1$$
$$= \{i_r(0),i_r(1),\ldots,i_r(Q-1),i_r(Q),\ldots,i_r(R-1)\}$$

(3)

where $r = 0,1,\ldots,S-1$; $m = 0,1,\ldots,Q-1$; and $n = 0,1,\ldots,p-1$.

As described in Observation (3), there are $S$ different sets of $R$ pairs of coefficient matrices. Sets $I_r$, for $r = 0,1,\ldots,S-1$, are these distinct index sets. As given in Observation (2), there are $R$ distinct pairs of coefficient matrices $A(i,j)$ and $B(i,j)$ used by the state variables $x(i,j)$ on each diagonal when they are related to the state variables on the previous diagonal. The index set $I_r$ gives the indices of $R$ pairs of coefficient matrices $A(i,k-i)$ and $B(i,k-i)$ used by the state variables $x(i,j)$ on $r$th diagonal when $k \equiv r \text{ mod } S$.

**Definition (2):** Permutation Matrices.

Let $T_m$ be an $m \times m$ cyclic permutation matrix such that $T_m[x_1,x_2,\ldots,x_{m-1},x_m]^T = [x_2,x_3,\ldots,x_m,x_1]^T$.

It is known [19] that $T_m$ has the following properties:

$$T_m^T T_m = I_{m \times m} \text{ or } T_m^{-1} = T_m^T;$$
$$[T_m^T x_1, x_2, \ldots, x_{m-1}, x_m]^T = [x_{i+1}, x_{i+2}, \ldots, x_{i-1}, x_i]^T;$$
$$T_m^m = I_{m \times m}.$$

Clearly, these properties are well preserved for the block cyclic permutation matrix $\tilde{T}_m = T_m \otimes I_{n \times n}$. That is, $\tilde{T}_m$ satisfies:

1) $T_m^m = I_{n \times n}$ or $T_m^{-1} = T_m^T$.
2) $([T_m^T x_1, x_2, \ldots, x_{m-1}, x_m]^T)^T = [x_{i+1}, x_{i+2}, \ldots, x_{i-1}, x_i]^T$.
3) $T_m^m = I_{n \times n}.$

**Definition (3):** Matrices $G_r^{(P)}$.

For $r = 0,1,\ldots,S-1$, define as shown in (4) at the bottom of the page where $p = 1,2,\ldots,p$ and $i_r(n) \in I_r$ (0 $\leq n \leq R - 1$) as defined (3).

It is easy to verify the following:

1) $G_r^{(P)} = (\tilde{T}_P)^{p} G_r (\tilde{T}_P)^{p}$.
2) $\prod_{r=S-1}^{0} G_r^{(P)} = (\tilde{T}_P)^{P} G_{S-1} (\tilde{T}_P)^{P} \prod_{r=S-1}^{0} G_r^{(P)}$
$$\times G_{S-2} (\tilde{T}_P)^{P} \cdots (\tilde{T}_P)^{P} G_0 (\tilde{T}_P)^{P} = (\tilde{T}_P)^{P} \prod_{r=S-1}^{0} G_r (\tilde{T}_P)^{P}.$$
The following definitions and observation are useful in deriving sufficient conditions for the stability of 2-D systems.

**Definition (4):** 2-D system stability.
A 2-D state-space system with state variable \(x(i,j)\) is asymptotically stable if \(\lim_{i,j \to \infty} x(i,j) = 0\).

**Definition (5):** 2-D energy function on diagonal.
For a 2-D system with state variable \(x(i,j)\), define the energy on the \(k\)th diagonal as

\[
U(k) = \sum_{l=0}^{k} |x(k-l,l)|,
\]

**Observation (4):** If
\[
\lim_{k \to \infty} U(k) = 0
\]
then the system is stable.

Observation (4) is true because if the energy on the progressing diagonals goes to zero, then the state variable \(x(i,j)\) must also approach zero in any direction in the \((i,j)\) plane.

### IV. Stability of the 2-D PSV System

In this section, a sufficient condition is first established in Theorem 1 for the stability of the 2-D PSV FM model. Then, a necessary condition and a different sufficient condition are given in Theorems 2 and 3, respectively, for the stability of 2-D PSV system.

**Theorem 1:** Consider the zero-input 2-D PSV FM state-space system

\[
x(i+1,j+1) = A(i+1,j+1)x(i+1,j) + B(i+1,j+1)x(i,j + 1)
\]

where the coefficients matrices \(A(i,j), B(i,j) \in \mathbb{R}^{L \times L}\) are periodically shift variant with period \((P,Q)\). The initial conditions are assumed such that \(x(i,j) = 0\), for \(i = 0\) or \(j = 0\), and \(x(i,0) = 0\), for \(i \geq 1\), \(j \geq 1\). For \(r = 0,1,2,\ldots,S-1\), define

\[
F_r = \max_{0 \leq n \leq \frac{r-1}{S-1}} \{ |A(i_r(n+1))| + |B(i_r(n))| \}
\]

where \(I_r\) are the index sets defined in (3). If \(\rho(\prod_{r=1}^{S-1} F_r) < 1\), then the system is asymptotically stable.

**Proof:** Using the system description and the initial conditions, the state variables on the \((k+1)\)-th diagonal can be written as

\[
x(k+1,0) = B(k+1,0)x(k,0),
\]

\[
x(k,1) = A(k,1)x(k,0) + B(k,1)x(k-1,1),
\]

\[
x(k-1,2) = A(k-1,2)x(k,1) + B(k-1,2)x(k-2,2),
\]

\[
\vdots
\]

\[
x(0,k+1) = A(0,k+1)x(0,k),
\]

The following inequality is obtained by taking the absolute value of both sides, summing up the above equations, and using the triangular inequity:

\[
\sum_{l=0}^{k+1} |x(k+1-l,l)|
\]

\[
\leq \sum_{l=0}^{k} \{|A(k-l,l+1)| + |B(k+1-l,l)|\}|x(k-l,l)|.
\]

(7)

Define

\[
U(k) = \sum_{l=0}^{k} |x(k-l,l)|
\]

Then (7) can be written as

\[
U(k+1) \leq F_{k+1} U(k)
\]

where

\[
F_{k+1} = \max_{0 \leq n \leq \frac{r-1}{S-1}} \{ |A(i_r(n+1))| + |B(i_r(n))| \}
\]

Using the recursion in (8), we have

\[
U(k+1) \leq F_0 F_1 \cdots F_k F_{k+1} U(k+1-S)
\]

\[
\leq (F_0 F_1 \cdots F_k F_{k+1} F_{k+1+1} \cdots F_{k+S+1}) \times U(k+1-S)
\]

\[
= \left( \prod_{z=0}^{0} F_z \right) \left( \prod_{z=0}^{S} F_z \right) \times \left( \prod_{z=0}^{S+S+1} F_z \right) \times U(k+1-S).
\]

If

\[
\rho \left( \prod_{r=1}^{S-1} F_r \right) < 1
\]

then clearly \(\lim_{k \to \infty} U(k) = 0\). \(\square\)

**Theorem 2:** Consider the 2-D zero-input PSV FM state-space system

\[
x(i+1,j+1) = A(i+1,j+1)x(i+1,j) + B(i+1,j+1)x(i,j + 1)
\]

where the coefficients matrices \(A(i,j), B(i,j) \in \mathbb{R}^{L \times L}\) are periodically shift variant with period \((P,Q)\). The initial con-
ditions are assumed such that \(x(i, j) = 0, i < 0 \) or \( j < 0 \), and \(x(0,0) = 0\). For \( i \geq 1, j \geq 1\), if

\[
\lim_{i+j \to \infty} x(i, j) = 0, \quad \gamma(r) = \left( \prod_{r=0}^{r=S-1} G_r \right) < 1,
\]

where \( r = 0, 1, \ldots, S-1, G_r \) is given by Definition (3), and \( T_p \) is the block permutation matrix defined in Definition (2) with \( m = p \) and \( n = QL \).

Proof: Consider the first quadrant of state variables \( x(i,j) \) where \( i+j = k+1 \). Let \((k+1) \equiv l \mod P\). Using the system description and the initial conditions, the state variables on the \((k+1)\)th diagonal can be written as

\[
x(k+1,0) = B(l,0)x(k,0), \\
x(k,1) = A(l-1,1)x(k,0) + B(l-1,1) \\
x(k,2) = A(l-2,2)x(k-1,1) \\
\vdots \\
x(k+1-R,R) = A(l,0)x(k-R+1,1, R-1) + B(l,0)x(k-R,R) \\
x(k+1-R-1,R+1) = A(l-1,1)x(k-R,R) + B(l-1,1) \\
\vdots \\
x(0,k+1) = A(0,l)x(0,k).
\] (9)

Since every \( R \)th state variable on the diagonal uses the same coefficient matrices, we add every \( R \)th state variables in (9) starting at \( x(k+1,0) \). For example, we add the state variables denoted by “**” in Fig. 1. Then we add every \( R \)th state variables starting with \( x(k,1) \). For example, we add all the state variables denoted by “\( \bullet \)“ in Fig. 1. Continuing this process, we get the \( R \) different equations. For \( k+1 = MR+N \), where \( M \) is a positive integer, and \( 0 \leq N \leq R-1 \), define

\[
t_0(k) = \begin{cases} M, & 0 \leq i \leq N-1 \\ M-1, & N \leq i \leq R-1 \end{cases} \\
t_0(k+1) = \begin{cases} M, & 0 \leq i \leq N \\ M-1, & N+1 \leq i \leq R-1 \end{cases}
\]

which are the total numbers of state variables on the \((k+1)\)th diagonal which use the same \( A(l-i,i) \) and \( B(l-i,i) \), respectively, when they relate to the state variables on the previous diagonal. Then the \( R \) different equations are

\[
\sum_{m=0}^{t_0(k)} x(k+1-Rm,Rm) \\
= A(l-0,0) \sum_{m=0}^{t_0(k)} x(k-Rm-R+1,Rm+R-1) \\
+ B(l-0,0) \sum_{m=0}^{t_0(k)} x(k-Rm,Rm)
\] (10)

\[
\sum_{m=0}^{t_1(k+1)} x(k+1-Rm-1,Rm+1) \\
= A(l-1,1) \sum_{m=0}^{t_1(k)} x(k-Rm,Rm) \\
+ B(l-1,1) \sum_{m=0}^{t_1(k)} x(k-Rm-1,Rm+1)
\] (11)

\[
\vdots
\]

\[
\sum_{m=0}^{t_{R-1}(k+1)} x(k+1-Rm-R+1,Rm+R-1) \\
= A(l+1,Q-1) \sum_{m=0}^{t_{R-2}(k)} x(k-Rm-R+2) \\
+ B(l+1,Q-1) \sum_{m=0}^{t_{R-2}(k)} x(k-Rm-R+1,Rm+R-1)
\] (12)

Let

\[
V(k+1) = \begin{bmatrix}
\sum_{m=0}^{t_0(k+1)} x(k+1-Rm,Rm) \\
\sum_{m=0}^{t_1(k+1)} x(k+1-Rm-1,Rm+1) \\
\vdots \\
\sum_{m=0}^{t_{R-1}(k+1)} x(k+1-Rm-R+1,Rm+R-1)
\end{bmatrix}
\]

Then (10)–(12) can be written as shown in the equation at the bottom of the page. That is

\[
V(k+1) = \hat{G}_l V(k)
\] (14)

where the matrix \( \hat{G}_l \) is shown at the bottom of the next page. Because of the periodicity of the coefficient matrices, \( \hat{G}_l \) repeats with period \( P \) as first index increases. Therefore

\[
V(k+1) = \begin{bmatrix}
B(l,0) & 0 & \cdots & 0 \\
A(l-1,1) & B(l-1,1) & \cdots & 0 \\
0 & 0 & \cdots & A(l+1,Q-1)
\end{bmatrix} V(k).
\]
(14) can be written as

\[
V(k+1) = \hat{G}_t \hat{G}_{i-1} \cdots \hat{G}_{i+1} V(k+1 - P) \\
= (\hat{G}_t \hat{G}_{i-1} \cdots \hat{G}_{i+1})(\hat{G}_t \hat{G}_{i-1} \cdots \hat{G}_{i+1}) \\
\times V(k+1 - 2P) \\
\vdots \\
= \left( \begin{array}{l} \hat{G}_z \\ \vdots \\ \hat{G}_z \\ \vdots \\ \hat{G}_z \end{array} \right) \left( \begin{array}{l} \hat{G}_z \\ \vdots \\ \hat{G}_z \\ \vdots \\ \hat{G}_z \end{array} \right)^{n-1} \left( \begin{array}{l} \hat{G}_z \\ \vdots \\ \hat{G}_z \\ \vdots \\ \hat{G}_z \end{array} \right) \\
\times V(k+1 - nP).
\]

(15)

Since the system is assumed stable, \( \lim_{k \to \infty} V(k) = 0 \). Thus the (15) implies that

\[
\rho \left( \prod_{i=P}^0 \hat{G}_t \right) < 1.
\]

Note that the matrices \( \hat{G}_t, \hat{G}_{S+1}, \cdots, \hat{G}_{S+P} \) use the same coefficient matrices with indices in \( I_r \) by shifting \( Q \) indices each other. So \( \prod_{i=P}^0 \hat{G}_t \) can be simplified further. Let \( l = P^S + r \) (\( P^S = 0,1, \cdots, P - 1 \)). Clearly, \( \hat{G}_t = \hat{G}_t^{(P)} \), where \( \hat{G}_t^{(P)} \) are defined in (4). Then \( \prod_{i=P}^0 \hat{G}_t = \prod_{i=P}^0 \hat{G}_t^{(P)} = (\prod_{i=S-1}^0 \hat{G}_t^{(P)})^{P} \) by the properties given in (5). Hence \( \rho (\prod_{i=P}^0 \hat{G}_t) < 1 \) if and only if \( \rho (\prod_{i=P}^0 \hat{G}_t^{(P)}) < 1 \).

**Theorem 3:** Consider the 2-D zero-input PSV FM state-space system

\[
x(i+1,j+1) = A(i+1,j+1)x(i+1,j) \\
+ B(i+1,j+1)x(i,j+1)
\]

where the coefficients matrices \( A(i,j), B(i,j) \in R^{L \times L} \) are periodically shift-variant with period \( (P,Q) \). The initial conditions are assumed such that \( x(i,j) = 0, i < 0 \) or \( j < 0 \), and \( x(i,0) = 0, x(0,j) = 0 \), for \( i \geq 0 \) or \( j \geq 0 \).

If \( \rho (\prod_{i=P}^0 \hat{G}_t^{(P)}) < 1 \), where \( \hat{G}_t^{(P)} \) are as defined in the Theorem 2, then \( \lim_{k \to \infty} x(i,j) = 0 \).

**Proof:** The proof follows an analysis similar to that used in the proof of Theorem 2, but absolute values are used in this case. We again consider the state variables on the diagonal \( i + j = k + 1 \), and the equations in (9). Take the absolute value of both sides of (10–12) and define

\[
W(k+1) = \left[ \begin{array}{c} \sum_{m=0}^{t_0(k+1)} |x(k+1 - Rm, Rm)| \\ \vdots \\ \sum_{m=0}^{t_n(k+1)} |x(k+1 - Rm - 1, Rm + 1)| \\ \vdots \\ \sum_{m=0}^{t_n(k+1)} |x(k+1 - Rm - R + 1, Rm + R - 1)| \end{array} \right].
\]

Using the triangular inequality, we have the inequality shown at the bottom of the next page.

That is

\[
W(k+1) \leq |\hat{G}_t| W(k).
\]

Since \( |\hat{G}_t| \) repeats with period \( P \) as the first index increases, the inequality in (16) can be written as

\[
W(k+1) \leq ||\hat{G}_t|| \hat{G}_{i-1} \cdots \hat{G}_{i+1} ||W(k+1 - P) \\
\leq (||\hat{G}_t|| \hat{G}_{i-1} \cdots \hat{G}_{i+1})(||\hat{G}_t|| \hat{G}_{i-1} \\
\cdots \hat{G}_{i+1}) ||W(k+1 - 2P) \\
\vdots \\
\times W(k+1 - nP).
\]

(17)

From (17), it is clear that if \( \rho (\prod_{i=S-1}^0 \hat{G}_t^{(P)}) < 1 \), then \( \lim_{k \to \infty} W(k) = 0 \). This in turn implies that if \( \rho (\prod_{i=S-1}^0 \hat{G}_t^{(P)}) < 1 \) then \( \lim_{k \to \infty} x(i,j) = 0 \).

In this section, two sufficient conditions for stability are established in Theorems 1 and 3, respectively. The condition given in Theorem 1 is more restrictive than the one in Theorem 3. (Please see Lemma 3 given in the Appendix for details.) However, the computation required to check the sufficient condition in Theorem 1 is significantly less than the one in Theorem 3 since the sizes of \( F_r \) are \( L \times L \) but the sizes of \( G_r \) are \( QL \times QL \). So, for a given 2-D PSV system, we will first use Theorem 1 to check the stability of the system. If the condition fails, we then check the necessary condition in Theorem 2. If the necessary condition fails, then we can conclude that the system is not stable; otherwise, we check the sufficient condition in Theorem 3. Examples given in the next section will illustrate this procedure.

**V. EXAMPLES**

In this section, three examples are given to illustrate the stability conditions established in the last section. The system in Example 1 is stable since it satisfies the sufficient condition given in Theorem 1. The system in Example 2 does not satisfy the sufficient condition in Theorem 1. It is not stable since it fails the necessary condition given in Theorem 2. In Example 3, the system fails the sufficient condition in Theorem 1 but satisfies the necessary condition. After checking the sufficient condition in Theorem 3, we can conclude the system is stable. For comparison purposes, we convert this system to an equivalent LSI system by using the method of [13] and check for stability.
**Example 1:** The system coefficients are given below with period \((2, 2)\):

\[
A(0, 0) = \begin{bmatrix} -0.1 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad A(1, 1) = \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 0.1 \end{bmatrix}, \quad A(1, 0) = \begin{bmatrix} 0.2 & 0.5 \\ -0.2 & 0.1 \end{bmatrix},
\]

\[
B(0, 0) = \begin{bmatrix} 0.1 & -0.2 \\ 0.4 & 0.3 \end{bmatrix}, \quad B(1, 1) = \begin{bmatrix} 0.1 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}, \quad B(1, 0) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & -0.2 \end{bmatrix}.
\]

In this case, \(p = 2, q = 2, p = 1, q = 1, S = 2\) and \(R = 2\). Definition (1) gives \(I_0 = \{(0,0), (1,1)\}\) and \(I_1 = \{(1,0), (0,1)\}\). From (6)

\[
F_0 = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.4 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.6 & 0.7 \\ 0.5 & 0.3 \end{bmatrix}.
\]

Since

\[
\rho\left(\prod_{r=1}^{0} F_r\right) = 0.9533 < 1
\]

the sufficient condition of Theorem 1 is satisfied and the system is stable.

**Example 2:** The system coefficients are given below with period \((2, 2)\):

\[
A(0, 0) = \begin{bmatrix} 0.1 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}, \quad A(1, 1) = \begin{bmatrix} -0.1 & 0.3 \\ -0.4 & 0.1 \end{bmatrix}, \quad A(1, 0) = \begin{bmatrix} 0.2 & 0.5 \\ -0.2 & 0.1 \end{bmatrix},
\]

\[
B(0, 0) = \begin{bmatrix} 0.8 & -0.2 \\ 0.4 & 0.3 \end{bmatrix}, \quad B(1, 1) = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}, \quad B(1, 0) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & -0.2 \end{bmatrix}.
\]

Since

\[
\rho\left(\prod_{r=1}^{0} F_r\right) = 1.9686 > 1
\]

the sufficient condition of Theorem 1 is not satisfied. Applying Theorem 2 to the system gives

\[
G_0 = \begin{bmatrix} 0.8 & -0.2 & 0.1 & 0.1 \\ 0.4 & 0.3 & 0.4 & -0.1 \\ -0.1 & 0.3 & -0.5 & 0.2 \\ -0.4 & -0.1 & 0.1 & 0.3 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.5 & 0 & -0.4 & -0.7 \\ -0.1 & -0.2 & -0.2 & 0.1 \\ 0.2 & 0.5 & -0.2 & -0.4 \\ -0.6 & 0.1 & 0.3 & 0.1 \end{bmatrix}
\]

and

\[
\rho\left(\prod_{r=1}^{0} G_r\right) = 1.1659 > 1.
\]

The necessary condition of Theorem 2 is not satisfied and hence the system is unstable.

**Example 3:** The system coefficients are given below with period \((2, 2)\):

\[
A(0, 0) = \begin{bmatrix} -0.6 & 0.2 \\ -0.4 & 0.1 \end{bmatrix}, \quad A(1, 1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad A(1, 0) = \begin{bmatrix} 1.2 & 0.3 \\ -0.4 & 0.1 \end{bmatrix}, \quad A(0, 1) = \begin{bmatrix} -0.2 & 0.4 \\ 0 & 0.3 \end{bmatrix},
\]

\[
B(0, 0) = \begin{bmatrix} 0.3 & 1 \\ 0.1 & -0.3 \end{bmatrix}, \quad B(1, 1) = \begin{bmatrix} -0.4 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}, \quad B(1, 0) = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.2 \end{bmatrix}.
\]

The conditions of Theorems 1, 2, and 3 give

\[
\rho\left(\prod_{r=1}^{0} F_r\right) = 2.0010 > 1
\]

\[
\rho\left(\prod_{r=1}^{0} G_r\right) = 0.4117 < 1
\]

\[
\rho\left(\prod_{r=1}^{0} [G_r]\right) = 0.9807 < 1.
\]

Since the sufficient condition of Theorem 3 is satisfied, the system is stable.

Using the method of [13] to convert this system to an equivalent LSI model, the resulting LSI matrices \(\tilde{A}^{(0)}\) and \(\tilde{A}^{(1)}\) are as follows:

\[
\tilde{A}^{(0)} = \begin{bmatrix} 0 & 0 & -0.6 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.84 & 0.27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.07 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.22 & 0.07 & 0 & 0 & -0.2 & 0.4 \\ 0 & 0 & -0.08 & 0.02 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & -0.338 & -0.111 & 0 & 0 & -0.02 & 0.1 \\ 0 & 0 & 0.048 & -0.018 & 0 & 0 & -0.08 & 0.19 \end{bmatrix}
\]

\[
\tilde{A}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0.3 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & -0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.39 & 0.03 & 0.4 & 0 \\ 0 & 0 & 0 & -0.11 & -0.07 & 0.1 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & -0.06 & 0 & 0 \\ 0 & 0 & 0 & -0.164 & -0.038 & -0.14 & 0.04 \\ 0 & 0 & 0 & -0.003 & -0.03 & -0.01 & 0.06 \end{bmatrix}
\]

\[
W(k+1) = \begin{bmatrix} 0 & B(L,0) & \cdots & 0 \\ 0 & A(L-1,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(L+Q-1,1) \end{bmatrix} W(k).
\]
For LSI systems, the conditions of Theorems 1 and 3 degenerate into the same condition. For this system, the condition yields $\rho \left( |A| + |A^T| \right) = 1.2119 > 1$. The condition is not satisfied and therefore no conclusion can be made about stability. This illustrates that the condition of Theorem 3 is less restrictive when applied directly to the PSV system.

VI. CONCLUSION

This paper studies the stability of 2-D PSV digital filters formulated in the FM state-space model. Conditions are established for the stability of the considered system. Two of these results are sufficient conditions and one is a necessary condition. One sufficient condition, Theorem 1, is more restrictive than the other, Theorem 3. However, the condition of Theorem 1 is simpler in computation and is easier to apply. The necessary condition of Theorem 2 can be used to quickly check for instability of a given system. Similarity transformation matrices may be used to transform the coefficient matrices in order to obtain an equivalent system. These coefficient matrices may then be optimized so that the sufficient conditions can be satisfied easier. This is a nontrivial problem and needs to be addressed in the future.

APPENDIX

A. Verifications for Observations (2) and (3)

Observation (2): The state variable $x(i, k-i)$ along the $k$th diagonal depends on the coefficient matrices $A(i, k-i)$ and $B(i, k-i)$ when it relates to the state variables on the $(k-1)$th diagonal. Assume that $A(i, k-i) = A(j, k-j)$ for some $0 \leq i, j \leq k$. Then $i \equiv j \mod P$ and $k - i \equiv k - j \mod Q$. That implies $i \equiv j \mod P$ and $i \equiv j \mod Q$. Therefore, $i \equiv j \mod R$. This is also true for matrices $B(i, j)$ since the indices for $A$ and $B$ are the same.

Observation (3): Consider the state variables on $k$th and $l$th diagonals where $l > k > R$. Assume that $A(i, l-i) = A(j, l-j)$. Then $i \equiv j \mod P$ and $i \equiv l \equiv j \mod Q$. In other words, there exist integers $\alpha$ and $\beta$ such that $j-i = \alpha P$ and $(l-k) - (j-i) = \beta Q$. So, $1 - k = \alpha P + \beta Q = \alpha Sp + \beta Sq = (ep + \beta q)S$. Thus, $l \equiv k \mod S$.

Lemmas 1 and 2 given in the following will be used to prove Lemma 3.

Lemma 1: Let $A = [A_{ij}], B = [B_{ij}]$ be in $R^{m \times mn}$ where $A_{ij}, B_{ij} \in R^{n \times n}$ and let $C = AB = [C_{ij}]$ where $C_{ij} \in R^{n \times \infty}$. If $\sum_{i=0}^{m-1} A_{ij} = E_A$ and $\sum_{i=0}^{m-1} B_{ij} = E_B$, then $\sum_{i=0}^{m-1} C_{ij} = E_A E_B$. For $j = 0, 1, \ldots, m-1$. Proof: $C = AB$ gives $C_{ij} = \sum_{k=0}^{m-1} A_{ik} B_{kj}$. Therefore $\sum_{i=0}^{m-1} C_{ij} = \sum_{i=0}^{m-1} \sum_{k=0}^{m-1} A_{ik} B_{kj} = \sum_{k=0}^{m-1} \left( \sum_{i=0}^{m-1} A_{ik} \right) B_{kj} = E_A \sum_{k=0}^{m-1} B_{kj} = E_A E_B$.

Lemma 2: Let $A = [A_{ik}]$ in $R^{m \times \infty}$ where $A_{ik} \in R^{n \times n}$. If $A \geq E$ and $\sum_{i=0}^{m-1} A_{ij} = E$ for $j = 0, 1, \ldots, m-1$, then $\rho(A) = \rho(E)$.

Proof: When $\eta = 1$, this lemma is equivalent to Theorem 8.1.21 given in [20].

Let $\lambda_{E,X}$ be an eigenvalue and the corresponding eigenvector of $E$. Then $EX = \lambda_{E,X}X$, or $(\sum_{i=0}^{m-1} A_{ij})X = \lambda_{E,X}X$ for $j = 0, 1, \ldots, m-1$. This implies $A Y = \lambda_{E,Y}$ where $Y = [X^T, X^T, \ldots, X^T]^T$. Hence $\lambda_{E,X}$ is also an eigenvalue of $A$ and then $\rho(A) \geq \rho(E)$.

Now, assume $\rho(A) > \rho(E)$. Then there must exist an eigenvalue $\lambda_{A,X}$ of $A$ such that $|\lambda_{A,X}| > |\lambda_{E,X}|$ for all eigenvalues of $E$. Let $Y_i$ be the corresponding eigenvector of $\lambda_{A,X}$ and $Y_i = [Y_{i0}^T, Y_{i1}^T, \ldots, Y_{in}^T]^T$ where $Y_i \in R_{n \times 1}$ and $A Y_i = \lambda_{A,X} Y_i$. Then, $\sum_{i=0}^{m-1} A_{ij} Y_{ij} = \lambda_{A,X} Y_{ij}$ for $i = 0, 1, \ldots, m-1$. Since

$$|\lambda_{A,X}| \sum_{i=0}^{m-1} |Y_{ij}| = \sum_{i=0}^{m-1} |A_{ij} Y_{ij}| = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} |A_{ij} Y_{ij}| \leq \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} |A_{ij}| |Y_{ij}| = E \sum_{j=0}^{m-1} |Y_{ij}|$$

we have $\rho(E) \geq |\lambda_{A,X}|$ by Corollary 8.1.29 given in [20]. This contradicts the assumption $\rho(E) < |\lambda_{A,X}|$. Thus $\rho(A) = \rho(E)$ has to be true.

The following lemma shows that the condition of Theorem 1 is more restrictive than that of Theorem 3.

Lemma 3: Consider $F_r$, $G_{s-1}$, and $T_p$ given in Theorems 1 and 3. If $\rho(\prod_{r=0}^{S-1} |G_r|) < 1$ then

$$\rho \left( \prod_{r=S-1}^{P} \left( \begin{array}{c} 0 \\ r=S-1 \end{array} |G_r| \right) \right) < 1.$$

Proof: Observe that $T_p \prod_{r=S-1}^{P} |G_{s-1}| = T_p \prod_{r=S-1}^{P} |G_{s-1}|$ and the block column sums of $T_p \prod_{r=S-1}^{P} |G_{s-1}|$ are the same as the ones of $|G_{s-1}|$. Observe also that matrix $[G_r]$ is constructed by matrices $[A(i, r+n)]$ and $[B(i, r+n)]$ for $i \in I_r$ and many zero matrices. The block column sums of $|G_r| \leq \max_{0 \leq n < k \in I_r} \text{with} \{\|A(i, r+n)\|, \|B(i, r+n)\|}$. For $r = 0, 1, \ldots, S-1$ such that all block column sums of $|G_r|$ are equal to $F_r$. This can be easily done by adding positive matrices to those block columns whose sums are not equal to $F_r$. Clearly, $|G_r| \leq |F_r|$ for $r = 0, 1, \ldots, S-1$. Then

$$\rho \left( \prod_{r=S-1}^{P} \left( \begin{array}{c} 0 \\ r=S-1 \end{array} |G_r| \right) \right) = \rho \left( T_p \prod_{r=S-1}^{P} \left( \begin{array}{c} 0 \\ r=S-2 \end{array} |G_r| \right) \right)$$

$$\leq \rho \left( \prod_{r=S-1}^{P} |G_r| \right).$$

Since each $|G_r|$ has a constant block column sum $F_r$, the block column sum of $\prod_{r=S-1}^{P} |G_r|$ is the same as the one for $\prod_{r=S-1}^{P} F_r$ by Lemma 1. Then by Lemma 2, the equality $\rho(\prod_{r=S-1}^{P} |G_r|) = \rho(\prod_{r=S-1}^{P} F_r)$ holds.

ACKNOWLEDGMENT

This paper was written in part while M.-Q. Chen was at the University of Colorado at Denver as a Visiting Professor.
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