Stability of a Shift-Variant 2-D State-Space Digital Filter

Glen W. Mabey, Tamal Bose
Center for High-speed Information Processing
Electrical and Computer Engineering Department
Utah State University, Logan, Utah 84322
USA
Email: Glen.Mabey@usu.edu, tbose@ece.usu.edu

Mei Chen
Department of Mathematics and Computer Science
The Citadel, Charleston, SC 29409
USA
Email: mei.chen@citadel.edu

Abstract—Sufficient conditions for stability of time-varying 1-D systems are already well established. This work treats the 2-D case in an approach that parallels that of the 1-D, yet at the same time reveals the heightened complexity of the extension. When “double exponential stability” is guaranteed for a certain set of homogeneous equations, the 2-D system is BIBO stable. The result applies to a generalized form of the Givone-Roesser state-space equations.

I. INTRODUCTION

Digital filter stability is a classic topic in signal processing. Stability of time-varying filters is much more difficult to establish than for time-invariant ones. Furthermore, 2-D systems pose their own challenges to guaranteeing stability, and generally the requirements are substantially more restrictive than the 1-D conditions. Shift-variant 2-D state space systems are consequently quite involved when the stability question is asked. A recent publication by the authors identified a somewhat restricted framework for the 2-D system which was stable [1]. The present work overcomes those limitations, and presents a larger space on which the conditions can be applied.

Outlining, we have in Section II a recap of the conditions for stability of a 1-D state-space time-varying system. Then in Section III we present the extension to 2-D and prove its usefulness. Finally, in Section IV, simulation of a system satisfying the conditions is run, with some concluding remarks.

II. THE GENERALIZED 1-D CONDITIONS

In Chapter 6 of [2], Regalia establishes conditions for stability on a generalized, time-varying, single-input–single-output, 1-dimensional state-space system

\[ x(n+1) = A(n)x(n) + b(n)u(n) \] (1)

\[ y(n) = c^T(n)x(n) + d(n)u(n). \] (2)

The proof in [2] is managed in a sequence of intermediate conditions, summarized here as C1–C5, which build on each other to provide the final result.

First, we have that \( y(n) \) in (2) remains bounded if:

C1 the input \( u(n) \) remains bounded for all time;
C2 the system parameters \( (A(n), b(n), c(n), d(n)) \) remain bounded for all time; and
C3 the state vector \( x(n) \) remains bounded for all time.

C1 is clearly the “bounded input” part of BIBO stability. C2 is assumed to be true. Consequently, the proof involves showing that C3 implies BIBO stability of (2), and then other conditions need to be established that guarantee C3.

To guarantee C3, it must hold that (1) be BIBO stable, which is assured with C4:

\[ x(n+1) = A(n)x(n) \]

is exponentially stable provided there exists a symmetric positive definite matrix \( P \) fulfilling a Lyapunov equation

\[ P = A^T(n)PA(n) = C(n)C^T(n) \]

such that the resulting sequence \( \{C(n)\} \) gives \( [A(n), C^T(n)] \) as a uniformly observable pair.

In this context, we define \( O(m, n) \) as the observability matrix constructed from the pair \( [A(n), C^T(n)] \) over the time window \( n, n+1, \ldots, m \):

\[ O(m, n) = \begin{bmatrix} C^T(n) \\ C^T(n+1)A(n) \\ \vdots \\ C^T(m)\Phi(m, n) \end{bmatrix}, \]

where \( \Phi(m, n) = \prod_{k=0}^{m-n-1} A(k) \). The pair \( [A(n), C^T(n)] \) is said to be uniformly observable if there exists an integer \( M \) and positive constants \( c_1 \) and \( c_2 \) such that

\[ 0 < c_1 I \leq O^T(n + M - 1, n)O(n + M - 1, n) \leq c_2 I < \infty, \]

for all \( n \).

Thus, assurance of C5 implies C4 which implies C3 which, with C2, implies BIBO stability of (2).

III. THE 2-D LINEAR IIR FILTER CONDITIONS

We now extend the result of Section II to a shift-variant, generalized version of the Givone-Roesser state space formulation for a 2-D linear IIR filter, as

\[ \begin{bmatrix} x^H(i+1, j) \\ x^V(i, j+1) \end{bmatrix} = A(i, j)x(i, j) + b(i, j)u(i, j) \] (3)

\[ y(i, j) = c^T(i, j)x(i, j) + d(i, j)u(i, j) \] (4)

and

\[ A(i, j) = \begin{bmatrix} A_{11}(i, j) & A_{12}(i, j) \\ A_{21}(i, j) & A_{22}(i, j) \end{bmatrix}, \]

\[ b(i, j) = \begin{bmatrix} b_1(i, j) \\ b_2(i, j) \end{bmatrix} \]
\[ e(i, j) = \begin{bmatrix} c_1(i, j) \\ c_2(i, j) \end{bmatrix}, \quad \mathbf{x}(i, j) = \begin{bmatrix} x^H(i, j) \\ x^V(i, j) \end{bmatrix}, \]

where \( A_{11}(i, j) \) and \( A_{22}(i, j) \) are square and the size of all other elements follows naturally.

A. Overview of Conditions

With this problem formulation, we have analogous preliminary conditions for stability:

C1' the input \( u(i, j) \) remains bounded for all indices;

C2' the system description \((A(i, j), b(i, j), c(i, j), d(i, j))\) remains bounded for all indices; and

C3' the state vector \( \mathbf{x}(i, j) \) remains bounded for all indices.

That these three ensure stability is proven in Section III-B. The next two requirements are not nearly so straightforward in their translation to 2-D, and together they embody the contribution of this work. First we give some definitions.

Definition 1: For the system

\[
\begin{bmatrix} x^H(i+1, j) \\ x^V(i+1, j) \end{bmatrix} = \begin{bmatrix} A_{11}(i, j) & A_{12}(i, j) \\ A_{21}(i, j) & A_{22}(i, j) \end{bmatrix} \begin{bmatrix} x^H(i, j) \\ x^V(i, j) \end{bmatrix},
\]

(5)

define a path between two points \( p_2 \) and \( p_1 \) to be a sequence of matrices \( \{ A^*(p_2 - 1), A^*(p_2 - 2), \ldots, A^*(p_1) \} \) the product of which \( \prod_{k=1}^l A^*(p_2 - k) \) is one of the terms by which \( x^*(p_1) \) influences the calculation of \( x^*(p_2) \), where \( A^* \in \{ A_{11}, A_{12}, A_{21}, A_{22} \} \) and \( x^* \in \{ x^H, x^V \} \).

By \( (p_2 - 1) \) we mean one of \( \{(i - 1, j), (i, j - 1)\} \), and by \( (p_2 - 2) \) we mean one of \( \{(i - 2, j), (i - 1, j - 1), (i, j - 2)\} \).

A clearer interpretation of this definition as well as its utility and application will become apparent in the development of the proof.

Definition 2: We coin the term double exponential stability, which pertains to a path, meaning that the homogeneous equation \( \hat{x}(p) = \prod_{i=1}^l A^*(p - k) \hat{x}(p - z) \) satisfies

\[ ||\hat{x}(p)|| \leq \beta \cdot \gamma^l ||\hat{x}(p - z)|| \]

for all \( z \in \mathbb{N} \), given a bounded value for \( \hat{x}(p - z) \), with \( 0 \leq \gamma < 0.5, \beta \geq 1 \), and \( \gamma, \beta \) are constants.

To guarantee C3*, it must hold that (3) be BIBO stable, which is assured with C4* (proven in Section III-D).

C4* Every path between any two points \( p_1 \) and \( p_2 \) is double exponentially stable.

In order to apply C5*, which ensures C4*, we now require that \( A_{11} \) and \( A_{22} \) are of the same size.

C5* A path \( \{ A^*(\cdot) \} \) is double exponentially stable if there exists a symmetric positive definite matrix \( P \) fulfilling a Lyapunov equation

\[ P - 2A^* (p_k) P [2A^* (p_k)]^T = C(k) C^T(k) \]

such that the resulting sequence \( \{C(\cdot)\} \) gives \( \{2A^*(\cdot), C^T(\cdot)\} \) as a uniformly observable pair.

B. C3* Implies BIBO stability

Consider the magnitude of the filter output \( \hat{y}(i, j) \) as given by

\[ ||\hat{y}(i, j)|| = \left\| c^T(i, j) \mathbf{x}(i, j) + d(i, j) u(i, j) \right\| \]

\[ = \left\| \begin{bmatrix} c^T(i, j) & d(i, j) \end{bmatrix} \begin{bmatrix} \mathbf{x}(i, j) \\ u(i, j) \end{bmatrix} \right\| \]

\[ \leq \left\| \begin{bmatrix} c^T(i, j) \\ d(i, j) \end{bmatrix} \right\| \left\| \begin{bmatrix} \mathbf{x}(i, j) \\ u(i, j) \end{bmatrix} \right\|. \]

If C2* holds, then \( ||[c^T(i, j) \quad d(i, j)]|| \) will remain bounded. If C1* also holds, then the boundedness of \( \hat{y}(i, j) \) depends exactly on \( ||\mathbf{x}(i, j)|| \) being bounded.

C. Exploring the Problem

Consider this expanded form for the state update equation (3):

\[
x^H(i + 1, j) = A_{11}(i, j) x^H(i, j) + A_{12}(i, j) x^V(i, j) + b_1(i, j) u(i, j)
\]

(6)

\[
x^V(i + 1, j) = A_{21}(i, j) x^H(i, j) + A_{22}(i, j) x^V(i, j) + b_2(i, j) u(i, j).
\]

(7)

Now (6) can be rewritten as

\[
x^H(i, j) = A_{11}(i - 1, j) x^H(i - 1, j) + A_{12}(i - 1, j) x^V(i - 1, j) + b_1(i - 1, j) u(i - 1, j),
\]

and similarly for (7). Then these two equations may be substituted back into (6). When this procedure is iterated again, the rearranged result appears as Algorithm 1. In the 1-D case, when this type of procedure is carried out, each \( x(k) \) term is only included once in the calculation of \( x(l) \) where \( l > k \). However, the terms \( x^H(i - 1, j - 1) \) and \( x^V(i - 1, j - 1) \) are each included twice in the computation of \( x^H(i + 1, j) \). Equally significant is that the term \( u(i - 1, j - 1) \) is included twice. Then there are two paths between \( (i - 1, j - 1) \) and \( (i, j) \). Problematically, the number of paths between \( p_2 \) and \( p_1 \) has a combinatorial relationship, which forms a rotated version of Pascal’s triangle. Given the combinatorial expression for the value of an element of Pascal’s triangle, we have that the number of paths, \( h \), from point \( (i - m, j - n) \) to \( (i + 1, j) \) is

\[
h = \binom{m + n}{m} = \frac{(m + n)!}{m!n!}.
\]

(8)

A subset of the resulting values along with their indices are shown in Figure 1.
Algorithm 1 The expression for $x^H(i + 1, j)$ extended backwards two steps.

\[
x^H(i + 1, j) = A_{11}(i, j)x^H(i, j) + A_{12}(i, j)x^V(i, j) + b_1(i, j)u(i, j)
\]
\[
+ A_{11}(i, j)A_{11}(i - 1, j)A_{12}(i - 2, j)x^H(i - 2, j)
\]
\[
+ [A_{11}(i, j)A_{12}(i - 1, j)A_{21}(i - 1, j - 1) + A_{12}(i, j)A_{21}(i, j - 1)]x^H(i - 1, j - 1)
\]
\[
+ [A_{11}(i, j)A_{12}(i - 1, j)A_{22}(i - 1, j - 1) + A_{12}(i, j)A_{22}(i, j - 1)]x^V(i - 1, j - 1)
\]
\[
+ A_{12}(i, j)A_{22}(i, j - 1)A_{21}(i, j - 2)x^H(i, j - 2)
\]
\[
+ A_{11}(i, j)A_{22}(i, j - 1)A_{22}(i, j - 2)x^V(i, j - 2)
\]
\[
+ A_{11}(i, j)A_{11}(i - 1, j)b_1(i - 2, j)u(i - 2, j)
\]
\[
+ [A_{11}(i, j)A_{12}(i - 1, j)b_2(i - 1, j - 1) + A_{12}(i, j)A_{21}(i, j - 1)b_1(i, j - 1)]u(i - 1, j - 1)
\]
\[
+ A_{12}(i, j)A_{22}(i, j - 1)b_2(i, j - 2)u(i, j - 2)
\]
\[
+ A_{11}(i, j)b_1(i, j - 1)u(i, j - 1) + A_{12}(i, j)b_2(i, j - 1)u(i, j - 1) + b_1(i, j)u(i, j)
\]

Algorithm 2 The expression for $x^H(i + 1, j)$ extended backwards to $-\infty$.

\[
x^H(i + 1, j) = \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( \frac{m+n}{m} \right) \Phi^*((i + 1, j), (m, j - n))x^H(i - m, j - n)
\]
\[
+ \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( \frac{m+n}{m} \right) \Phi^*((i + 1, j), (m, j - n))x^V(i - m, j - n)
\]
\[
+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m+n}{m} \right) \Phi^*((i + 1, j), (m, j - n))b^*(i - m, j - n)u(i - m, j - n),
\]

This process can be pursued until $x^H(i + 1, j)$ has been written as a function of $\{x^H((-, -\infty)), \{x^H(-\infty, -\infty)\}, \{u(\cdot, \cdot)\}$. We attempt to write this expression, but it is somewhat difficult to do so in a clear and concise way, due to the way the expansion occurs. Accordingly, we define $\Phi^*(p_2, p_1)$ to be one of the product terms involving $A_{11}, A_{12}, A_{21},$ and $A_{22}$ which compose a path from $p_1$ to $p_2$, such as $A_{11}(i, j)A_{12}(i - 1, j)A_{21}(i - 1, j - 1)$ and $A_{12}(i, j)A_{21}(i, j - 1)A_{11}(i - 1, j - 1)$ are for $(i - 1, j - 1)$ to $(i + 1, j)$. When $p_1 = (i - m, j - n)$ and $p_2 = (i + 1, j)$, there are $m + n + 1$ terms in any $\Phi^*(p_2, p_1)$. The poor notation enters when we desire, for example, to indicate that there are two paths from $(i - 1, j - 1)$ to $(i + 1, j)$, which we write as $2 \cdot \Phi^*((i + 1, j), (i - 1, j - 1))$. This lack of precision is forgivable in the context of a supremum on the norm of these terms, as shall be shown later. As this expansion is carried out, the result is Algorithm 2. Again, to be concise, we use $b^*$ to indicate either $b_1$ or $b_2$.

It can be shown that the combinatorial function can be bounded as in the following lemma.

Lemma 1: Given $m, n \in \mathbb{W}$,

\[
\left( \frac{m+n}{m} \right) \leq 2^{(m+n)}.
\]

The proof of this lemma, while not tremendously technical, is too lengthy to be included in this publication, but will be included in a later one.

D. C4* Implies C3*

For notational brevity, define $p_1 = (i - m, j - n)$ and $p_2 = (i + 1, j)$. Now, define $\varphi(p_2, p_1) = \{\Phi^*(p_2, p_1)\}$ and $\Lambda(p_2, p_1) = \sup_{\varphi(p_2, p_1)} \|\Phi^*(p_2, p_1)\|$, and impose the following constraint.

C4* (restated)

\[
\Lambda((i, j), (i - m, j - n)) \leq \beta \gamma^{(m+n)}
\]

where $0 \leq \gamma < 0.5$, $\beta \geq 1$, and $\gamma, \beta$ are constants.

Then, the norm of the first term of Algorithm 2 is

\[
\left\| \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( \frac{m+n}{m} \right) \Phi^*(p_2, p_1) x^H(p_1) \right\|
\]

\[
\leq \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( \frac{m+n}{m} \right) \Lambda(p_2, p_1) \left\| x^H(p_1) \right\|
\]

\[
\leq \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( (2\gamma)^{(m+n)} \right) \left\| x^H(p_1) \right\|
\]

\[
= \beta \lim_{m \to \infty} \sum_{n=0}^{\infty} \left( (2\gamma)^{(m+n)} \right) \left\| x^H(i - m, j - n) \right\|
\]

\[
= \beta \cdot \sup_{n \in \mathbb{W}} \| x^H(-\infty, j - n) \| \lim_{m \to \infty} \left( (2\gamma)^{(m+n)} \right) \sum_{n=0}^{\infty} (2\gamma)^n
\]

\[
= \beta \cdot \sup_{n \in \mathbb{W}} \| x^H(-\infty, j - n) \| \cdot 0 \cdot \frac{1}{1 - 2\gamma}
\]

\[
= 0,
\]

since $0 \leq 2\gamma < 1$, if the initial conditions of the system, $\{x^H(-\infty, -\infty)\}$, are initialized to be bounded.

A similar progression for the norm of the second term of Algorithm 2,

\[
\left\| \lim_{n \to \infty} \sum_{m=0}^{\infty} \left( \frac{m+n}{m} \right) \Phi^*(p_2, p_1) x^V(p_1) \right\|
\]
that it too goes to zero. Now, define $Z = \sup \|b^*(p)\| \cdot \sup \|u(p)\|$, which (by C1* and C2*) is bounded. Then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{m+n}{m} \Phi^*(p_2, p_1) b^*(p_1) u(p_1) \leq Z \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \cdot \Lambda(p_2, p_1) \leq Z \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \cdot \beta \gamma(m+n) \leq Z \cdot \beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2\gamma)^{(m+n)}.$$ 

We now encounter the aspect of a double infinite summation, whereas in the 1-D case, there was only a single such summation. Accordingly, we present another lemma.

**Lemma 2:** For $\alpha < 1$,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha^{m+n} = \frac{1}{1-\alpha}.$$

**Proof:** Consider

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha^{m+n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha^m \alpha^n = \sum_{m=0}^{\infty} \alpha^m \sum_{n=0}^{\infty} \alpha^n = \sum_{m=0}^{\infty} \alpha^m \frac{1}{1-\alpha} = \frac{1}{(1-\alpha)^2}.$$ 

Thus the norm of Algorithm 2 yields

$$\left\| x^H(i + 1, j) \right\| \leq \sup \|b^*(p)\| \cdot \sup \|u(p)\| \cdot \beta \cdot \frac{1}{(1-2\gamma)^2} < \infty,$$

so when C1*, C2*, and C4* hold, then $x^H(i + 1, j)$ remains bounded\(^1\), and hence $y(i, j)$ is BIBO stable.

**E. C5* Implies C4*\(^1\)**

We remark on the tremendous similarity of form that C4* has to exponential stability; the only difference is the requirement on the base value. This similarity becomes the key point in finalizing the proof.

**Theorem 1:** The homogeneous system

$$x(n + 1) = A(n)x(n)$$

is double exponentially stable if and only if

$$x(n + 1) = 2A(n)\hat{x}(n)$$

is exponentially stable. That is, that

$$\|\hat{x}(m)\| \leq \beta \cdot \alpha^{m-n} \cdot \|\hat{x}(n)\|$$

for all $m \geq n$, where $0 \leq \alpha < 1$ and $\beta \geq 1$.

\(^1\)A nearly identical proof should be acknowledged which establishes that $x^H(i + 1, j)$ similarly remains bounded.

**Proof:** Define

$$\tilde{\Phi}(m, n) = \begin{cases} 1 & \text{if } m = n, \\ 2A(m - 1) \cdot 2A(m - 2) \cdots 2A(n) & \text{otherwise.} \end{cases}$$

As such, $\tilde{\Phi}(m, n) = 2^{m-n} \Phi(m, n)$. For sufficiency, take norms of $x(m) = \Phi(m, n)x(n)$ to obtain

$$\|x(m)\| = 2^{(m-n)} \|\Phi(m, n)x(n)\| \leq 2^{(m-n)} \|\Phi(m, n)\| \cdot \|x(n)\| \leq 2^{(m-n)} \beta \cdot \alpha^{m-n} \|x(n)\| = \beta \cdot \left(\frac{1}{2}\right)^{m-n} \|x(n)\|,$$

which is simply the definition of double exponential stability. For necessity, take norms of $\hat{x}(m) = \tilde{\Phi}(m, n)\hat{x}(n)$ to obtain

$$\|\hat{x}(m)\| \leq 2^{m-n} \|\tilde{\Phi}(m, n)\| \cdot \|\hat{x}(n)\| \leq 2^{m-n} \beta \cdot \gamma^{m-n} \|\hat{x}(n)\| = \beta \cdot (2\gamma)^{m-n} \|\hat{x}(n)\|,$$

which is simply the definition of exponential stability.

The cumulative result is then assured as follows. C5* gives that every path is double exponentially stable by Theorem 1 and C5.

**IV. SIMULATION AND CONCLUSIONS**

As an exercise, we show the results of a simulation of a system which satisfies C5*. The great challenge in this endeavor is to satisfy the requirement that "every path" be double exponentially stable. Periodicity of the $A^*$ matrices makes this tractable, yet it remains complicated.

**Example 1:** Consider the following system definition:

$$A(i, j) = \begin{cases} \tilde{A} & \text{when } (i + j) \text{ is odd,} \\ A & \text{when } (i + j) \text{ is even,} \end{cases}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & \tilde{A} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0.2 & 0.1 \\ 0.25 & -0.15 \end{bmatrix},$$

$$A = \begin{bmatrix} A & A \\ A & A \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -0.3 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

By construction, every $\Phi^*$ forming a path between any two points in the space is either $AAAAA \cdots$ or $AAAAA \cdots$. We establish the compliance with Theorem C5* numerically, by establishing the upper and lower-boundedness of the observability matrix by examining the smallest eigenvalue (and hence the positive definiteness) of the matrices that result from the uniform observability condition.

We remark that the condition C5* is substantially more restrictive than the requirement in 1-D, which generally is to be expected when 2-D conditions are compared to their 1-D counterparts.

**REFERENCES**
