Models of Hyperbolic Geometry

October 23, 2011
Begin with a circle \( C \) in the Euclidean plane, and its interior \( H \), as shown in the figure above. The components of this geometry are as follows:

**Point:** Any interior point of circle \( C \).

**Line:** Any diameter of \( C \) or any arc of a circle orthogonal to \( C \) in \( H \).
**Distance:** If $A$ and $B$ are 2 points and $l$ represents a line passing through $A$ and $B$, let its end points on $C$ be $M$ and $N$, define the distance from $A$ to $B$ as the real number,

$$AB^* = \left| \ln \left( \frac{AM \cdot BN}{AN \cdot BM} \right) \right| = \ln(AB, MN)$$
Angle Measure: If \( \triangle ABC \) is any angle in \( \mathbb{H} \) consisting of two rays \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \), then consider the Euclidean rays \( \overrightarrow{BA'} \) and \( \overrightarrow{BC'} \) (refer to the figure below), that are tangent to \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \) and in the same direction. Define, \( m \angle ABC^* = m \angle A'B'C' \).
This geometry satisfies all axioms for absolute geometry. For example, to verify the axiom “two points determine a line”, consider a coordinate system where \( C \) is taken as the unit circle \( x^2 + y^2 = 1 \). To find the circles orthogonal to \( C \), we will consider the equation \( x^2 + y^2 + ax + by = c \) and the equivalent center/radius form \( (x - h)^2 + (y - k)^2 = r^2 \),
Poincaré’s Disk Model

\[ x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2 \]
\[ x^2 + y^2 - 2hx - 2ky = r^2 - h^2 - k^2. \]

By matching the coefficients, \( a = -2h \), \( b = -2k \) and \( c = r^2 - h^2 - k^2 \).

Now consider a circle \( D \) that is orthogonal to \( C \). Orthogonality means that the tangents of the circles at their point of intersection \( P \) are perpendicular and since the radii drawn to the point \( P \) are perpendicular to the tangents, the tangent of one circle must pass through the center of the other. Thus \( \triangle OPD \) is a right triangle.

So, \( OD^2 = OP^2 + PD^2 = 1 + r^2 \), where \( D = (h, k) \) is the center of \( D \) and \( r \) is its radius. So by distance formula, \( h^2 + k^2 = 1 + r^2 \), from above we have \( c = r^2 - h^2 - k^2 \) or \( h^2 + k^2 = r^2 - c \) \( \Rightarrow r^2 - c = r^2 + 1 \) \( \Rightarrow c = -1 \), general equation of the circle \( D \) is \( x^2 + y^2 + ax + by = -1 \).
Thus, as long as \( C \) is the unit circle, to find the line (in hyperbolic sense) passing through 2 given points \( A, B \), we only have to substitute the coordinates in the equation above and solve for \( x \) and \( y \).

Note, there are no solutions if the coordinates of \( A \) and \( B \) are proportional but that means that the points are collinear with \( O \) and the “line” is then the diameter of \( C \) passing through \( A \) and \( B \).
Example 1: Find the equation of the “line” $D$ in $H$ passing through the points $A(0.1, 0.3)$ and $B(−0.1, 0.7)$. Find its center and radius and sketch the graph.
Consider all points $P(x, y)$ for which $y > 0$, this is the upper half-plane. All such points will be considered as "points" in this model. The $x$-axis is not a part of this geometry.
Beltrami-Poincaré’s Half-Plane Model

**Point:** Any point \( P(x, y) \) such that \( y > 0 \).

**Line:** Semicircles of the form \((x - a)^2 + y^2 = r^2, y > 0\) with center on the \( x \)-axis or vertical rays of the form \( x = a \) (constant), \( y > 0 \).

**Distance:** For any 2 “points” \( A \) and \( B \) and \( l \) the “line” containing \( A \) and \( B \), let the end points of \( l \) be (if it is a semicircle with equation as above) \( M(a + r, 0) \) and \( N(a - r, 0) \) or if \( l \) is a vertical ray, let its end point be \( M(a, 0) \), as shown below. Then in the 2 cases for \( l \), define “distance” from \( A \) to \( B \) as

\[
AB^* = | \ln \left( \frac{AM \cdot BN}{AN \cdot BM} \right) | = \ln(AB, MN),
\]

\[
AB^* = | \ln \frac{AM}{BM} |
\]
**Note**: the second distance formula given above is a special case of the first. This is because, we can see by using limit techniques that as $N$ recedes along the $x$-axis to $-\infty$ (i.e., $BN \to \infty$), the semicircle becomes a vertical ray and the ratio $\frac{AM \cdot BN}{AN \cdot BM}$ converges to $\frac{AM}{BM}$.

**Angle Measure**: Let $m\angle ABC^* = m\angle A'BC'$ where $\overrightarrow{BA'}$ and $\overrightarrow{BC'}$ are the Euclidean rays tangent to the “sides” of $\angle ABC$, as in the disk model.

**Note**: A semi-circular line in this model has an equation $x^2 + y^2 + ax = b$ (this can be deduced in the same way the equations of lines in the Disk Model was found).
Beltrami-Poincaré’s Half-Plane Model

Show geometrically that any 2 given h-points $A$ and $B$, there is always a unique h-line passing through them.

**Proof:** If $A, B$ lie on a vertical line $x = a$, then no semicircle centered on the $x$-axis can pass through $A$ and $B$, and the h-line $x = a, y > 0$, is the only one that passes through $A$ and $B$. On the other hand suppose $\overrightarrow{AB}$ is not a vertical line. Then the perpendicular bisector of Euclidean segment $\overline{AB}$ will be nonhorizontal, hence will meet the $x$-axis at a unique point $L$. Take $L$ as center and $AL$ as radius, and draw the semicircle through $A$ and $B$. This will be the unique h-line passing through $A$ and $B$ in this case.
Examples

**Example 2:** Find the equation of the h-line passing through the points $A(1, 3)$ and $B(1, 6)$

**Example 3:** **Hyperbolic Distance in Upper Half-Plane Model**
Find the h-distance $KW^*$ if $K = (1, 4)$ and $W = (8, 3)$. 
Validity of axioms in this model

The incidence axioms that apply to the plane are the following,

(a) Two points determine a line,
(b) There are 3 noncollinear points and each line contains at least 2 points.
((a) was proved as Example 1 and (b) is valid by construction.)

Next the distance (metric axioms),
(c) For any 2 h-points $A$ and $B$, $AB^* \geq 0$, with equality only when $A = B$.
(d) $AB^* = BA^*$.
(e) Ruler Postulate.

By definition of distance (c) is true, and when $A = B$, the ratio reduces to 1 and as $\ln 1 = 0$ so $AB^* = 0$.

For (d) note that,

$$AB^* = |\ln(AB, MN)| = |\ln \left( \frac{AM \cdot BN}{AN \cdot BM} \right)| = |\ln \left( \frac{BM \cdot AN}{BN \cdot AM} \right)^{-1}| =$$

$$|\ln(BA, MN)^{-1}| = |-\ln(BA, MN)| = |\ln(BA, MN)| = BA^*.$$
Validity of axioms in this model

For the Ruler Postulate, let $l = \overline{AB}$ be any h-line and $P$ be any point on $l$. Then assign the real number $x$ to $P$ so that $x = \ln(AP, MN)$, $(x = \ln(PM/AM)$, if $l$ is a vertical ray). Note that we will consider these expressions without the absolute values, this means that $x$ may be negative and thus the range of values of $x$ is the complete set of real numbers. First note that if $P \neq Q$, then $x \neq y$, since if $x = y$, then $(AP, MN) = (AQ, MN)$, which means $\frac{PN}{QN} = \frac{PM}{QM}$, but that is impossible, check the figure below.

Therefore, coordinates of each point is distinct.
Next we will prove that $PQ^* = |x - y|$. 

Note, $|x - y| = |\ln(AP, MN) - \ln(AQ, MN)| = |\ln\left(\frac{AP, MN}{AQ, MN}\right)| = 
|\ln\left(\frac{AM, PN}{AM, QN}\right)\frac{(AN, PM)}{(AN, QM)}| = |\ln\left(\frac{PN, QM}{PM, QN}\right)| = PQ^*$
Finally we will show the Plane Separation axiom.

Let \( l \) be any h-line and let the 2 sides of \( l \), \( H_1 \) and \( H_2 \) be as shown in the figure below. Since the convexity of \( H_1 \) and \( H_2 \) is clear when \( l \) is a vertical line, so we will concentrate on the case when \( l \) is a semicircle.
In case one encounters a situation as in the figure above, where $A$ and $B$ are two points in $H_1$, then the question is whether $\overline{AB}^*$ (the circular arc) is in $H_1$. Consider the arc $AB$ of the semicircle centered on the $x$-axis containing $A$ and $B$, since two semicircles cannot be tangential to each other (they cannot meet at an angle of measure zero), they must cross over each other (otherwise they will have no points in common).
Therefore if arc $AB$ is not entirely in $H_1$, then there exists two intersection points $C$ and $D$ and we have the figure seen above. This gives us 2 h-points in common with h-lines $\overrightarrow{AB}$ and $l$ but this is a contradiction. Hence, $\overline{AB}^* \subseteq H_1$. 